Relative Functoriality and L-Functions via Relative Trace Formulae

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Thanks for the fate, for its quiet and peace.

Dedication

To my family.
致我的家人。

Abstract

According to the relative Langlands functoriality conjecture, an admissible morphism between the L-groups of spherical varieties should induce a functorial transfer of the corresponding local and global automorphic spectra. Via the relative trace formula approach, two basic problems are the local transfer and the fundamental lemma on the geometric side of the relative trace formulae. In the first part of the thesis, we consider the rank one spherical variety case, where the admissible morphism between the L-groups is the identity morphism, in which case, Y. Sakellaridis has already established the local transfer ([Sak21]). We formulate the statement of the fundamental lemma for the general rank one spherical variety case and prove the fundamental lemma for the rank one spherical varieties of classical types.

In the second part of the thesis, we implement a Beyond Endoscopy approach to the functional equation of standard L-functions of GL_2, following the program outlined in [Sak23a]. We provide a direct derivation of a Poisson summation formula for the Kuznetsov trace formula based on the local Hankel transforms established in [Jac03]. By employing non-standard test functions, we develop a trace formula argument that yields a self-contained proof of the functional equation for standard L-functions of GL_2.

Contents

Chapter 1

Introduction

Tracking back to the early nineteenth century, P.G.L. Dirichlet introduced L-series in his seminal work [Dir37] to investigate the distribution of primes in arithmetic progressions. Shortly thereafter, in 1859, B. Riemann established the fundamental analytic properties of the Riemann zeta function in [Rie59], laying the groundwork for modern analytic number theory. Nearly a century later, this idea was further developed by K. Iwasawa and J. Tate and was reformulated in Tate's landmark thesis [Tat67] in the language of adeles using the theory of local and global harmonic analysis. This approach was subsequently generalized to the case of standard L-functions for GL_n by R. Godement and H. Jacquet in [GJ72].

In the decade preceding the development of the adelic framework, E. Hecke constructed L-functions associated with classical modular forms in [Hec36]. His discovery of the connection between Fourier coefficients of modular forms and L-functions provided early evidence of a profound link between periods and special values of L-functions. The profound relation between periods and L-values also features prominently in the study of modular symbols. This connection was pioneered by Y. Manin in [Man72, Man73, Man05] and subsequently studied in the adelic framework in [Har87]. Further foundational development can be found in the collection work of G. Harder, R. Langlands, and M. Rapoport [HLR86], as well as in the influential work of M. Harris [Har94] and S. Kudla [Kud04]. Concurrently, J.-L. Waldspurger [Wal85] established a landmark formula relating toric periods of GL_2 to special L-values, which was later reproved by H. Jacquet [Jac86]. This relationship was significantly expanded by B.H. Gross and D. Prasad [GP92, GP94], who formulated a series of conjectures extending this idea to higher rank orthogonal groups. These conjectures were later refined by A. Ichino and T. Ikeda [II10] and were proved by D. Ginzburg, D. Jiang, and S. Rallis [GJR04, GJR05], and D. Jiang and L. Zhang [JZ20] using automorphic descent. Recently, W. T. Gan, B.H. Grass and D. Prasad [GGP12] generalized the conjecture to classical groups. The unitary case has now been established through a monumental series of works, initiated by the trace formula framework formulated by H. Jacquet, S. Rallis [JR11]. This program was propelled forward by the proof of the fundamental lemma, first achieved by Z. Yun [Yun11] using a geometric method, and subsequently by R. Beuzart-Plessis [BP21] based on the smooth transfer proved by W. Zhang [Zha14]. Most recently, the fundamental lemma for the unit elements of the Jacquet-Rallis trace formula has been further extended to the full Hecke algebra by S. Leslie [Les23] and G. Wang and Z. Zhang [WZ26] using different methods. The full proof of the unitary Gan-Gross-Prasad conjecture was ultimately completed through the sustained efforts of R. Beuzart-Plessis, P.-H. Chaudouard, Y. Liu, Z. Michal, X. Zhu, and W. Zhang [BPLZZ21, BPCZ22, BPC25]. For another generalization of Waldspurger's results, see [Guo96, Zha15, FMW18, Cha20, Li21, Li22, Li23]. Regarding the relationship between linear periods, Shalika periods, and exterior square L-functions, one may refer to the foundational work of D. Bump, S. Friedberg, and H. Jacquet and J. Shalika [JS90a, BF90, FJ93]. One may also refer to the work of S. Leslie, J. Xiao, and W. Zhang [Les25, LXZ25a, LXZ25b] for the unitary variants. Furthermore, the study of Ginzberg-Rallis periods has provided a vital link to the study of exterior cube L-functions, as explored in the work of D. Ginzburg and S. Rallis in [GR00] and a recent work of C. Wan and L. Zhang [WZ23] for the unitary case.

Shortly following the work of E. Hecke, R.A. Rankin [Ran39], and A. Selberg [Sel40] independently developed a method to represent the L-functions of modular forms as integrals involving Eisenstein series, a technique now known as the Rankin-Selberg method, thereby facilitating the proof of the analytic continuation and functional equations of Rankin-Selberg L-functions. The Rankin-Selberg method was later systematically extended to general linear groups of arbitrary size in the seminal works of J. Cogdell, H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika [JPSS81, JS90b, CPS04]. Subsequently, a significant generalization known as the doubling method was introduced by

I.I. Piatetski-Shapiro and S. Rallis in [PSR84]. This method provided a powerful alternative for representing L-functions as integrals over a doubled group, circumventing the need for the existence of the Whittaker models. This framework was later extended to a wider class of groups. Notable developments include, but are not limited to, the work of D. Jiang [Jia94], E. Lapid and S. Rallis [LR05], as well as W. T. Gan [Gan12], and S. Yamana [Yam14].

One must also note the pivotal contributions of G. Shimura in [Shi75], whose approach, often referred to as the Shimura method, provided another powerful mechanism for studying the analytic behavior of L-functions. This theory was further developed by S. Gelbart and I.I. Piatetski-Shapiro [GPS78], providing the foundational work for the metaplectic correspondence. Moreover, the Shimura method was successfully applied to the study of symmetric square L-functions in the influential work of S. Gelbart and H. Jacquet [GJ78], D. Bump and D. Ginzburg [BG92], and more recently by S. Takeda [Tak14].

In his 1970 program, R.P. Langlands conjectured in [Lan70] that it should be possible to attach to each irreducible automorphic representation π of a reductive group G, and each finite dimensional representation ρ of its L-group ${}^{L}\mathrm{G}$, and L-function $L(s,\pi,\rho)$. These L-functions are defined via the Euler product of local factors, at least at unramified places, and are conjectured to admit a meromorphic continuation to the entire complex plane and satisfy a functional equation. Shortly thereafter, R.P. Langlands introduced the principle of functoriality in [Lan70, Lan97], a far-reaching conjecture which posits that to every L-homomorphism of L-groups, there corresponds a transfer of automorphic representations. Once established, this principle would imply that automorphic L-functions can be realized as standard L-functions of general linear groups. Consequently, the fundamental analytic properties would follow directly as a consequence of the foundational results established by R. Godement and H. Jacquet [GJ72].

The development of symmetric power functoriality for GL_2 represents a seminal sequence of advancements in the functoriality conjecture. This progress began with the work of S. Gelbart and H. Jacquet [GJ78], who established the functoriality of the symmetric square lift of GL_2 by utilizing the converse theorem for GL_3. This foundational result was subsequently extended to the symmetric cube by H. Kim and

F. Shahidi [KS02] and the symmetric fourth by H. Kim [Kim03]. Beyond the symmetric power case, several significant instances of functoriality have been established, including the tensor product of $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ by D. Ramakrishnan [Ram00], $\mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$ by H. Kim and F. Shahidi [KS02], and the exterior square of GL_4 by H. Kim [Kim03]. While a general proof of the functoriality principle remains elusive, substantial evidence has also been gathered through the theory of automorphic descent. This backward approach, which constructs transfer from general linear groups to classical groups, was initiated by A. Andrianov [And79], I.I. Piatetski-Shapiro ([PS83]), and M. Eichler and D. Zagier [EZ85]. The modern framework of the descent was reinvigorated by D. Ginzburg, S. Rallis, and D. Soudry in [GRS11], and was further expanded with the development of the twisted automorphic descent by D. Ginzburg [Gin12] and D. Jiang and L. Zhang [Jia14, JZ17, JZ20, JZ21].

Another landmark achievement in the realization of the functoriality principle is the theory of Endoscopy, introduced by R.P. Langlands [Lan79, Lan83], with the twisted extension detailed by R.E. Kottwitz and D. Shelstad [KS99]. One of the most significant triumphs of the trace formula is the confirmation of the endoscopy structure and the weak functorial transfer from quasi-split classical groups to general linear groups. This monumental achievement, established by J. Arthur [Art13] for orthogonal and symplectic groups and by C.P. Mok [Mok15] for unitary groups, provides a classification of the discrete automorphic spectrum for these groups. These results rest upon decades of foundational breakthroughs, most notably the proof of the fundamental lemma by B.C. Ngô [Ngo10] based on the work of D. Kazhdan and G. Lusztig [KL88], G. Laumon [Ngo06, LN08], D. Shelstad [She82], J.-L. Waldspurger [Wal91, Wal06, Wal08] and others.

With the Endoscopy program nearing completion, the field's focus has shifted to the Beyond Endoscopy program. This direction was prefigured in the thesis of Z. Rudnick [Rud90] and formally articulated by R.P. Langlands in [Lan04]. Early success was achieved by A. Venkatesh [Ven04], who utilized the Kuznetsov trace formula to establish the functorial transfer from tori to GL_2. Subsequently, a framework combining the trace formula with the Poisson summation formula was advanced by E. Frenkel, R.P. Langlands, and B.C. Ngô in [FLN10, FN11]. This approach was informed by the resolution of the fundamental lemma and the geometric Langlands program, with Langlands further refining these ideas for SL_2 in [Lan13], a case generalized in the thesis of D. Johnstone [Joh17]. Simultaneously, a series of papers by P. Hermann [Her11, Her12] also used the Kuznetsov trace formula to give a new proof of the analytic continuation and functional equation of standard L-functions and Rankin-Selberg L-functions. As noted in [Sak19, Section 6], Hermann's techniques bear a resemblance to Jacquet's formula in [Jac03]. However, certain aspects of Hermann's arguments were later deemed unsatisfactory. These issues were subsequently addressed in a recent work [KL25b] and followed up in another work [KL25a], where the authors adjusted some of the local assumptions. In parallel, Y. Sakellaridis [Sak13a, Sak19] began integrating elements of the relative Langlands program into the Beyond Endoscopy program. Significant progress in isolating the contribution of special representations for GL_2 from the Arthur-Selberg trace formula was made by S. Ali Altug [Alt15, Alt17, Alt20], who showed that the standard L-functions do not pick up functorial lifts in the cuspidal spectrum. Most recently, D. Johnstone and Z. Luo [JL20] have extended the consideration to the symmetric power transfer of GL_2.

To effectively insert L-functions into the trace formula, one may replace the unit element of the Hecke algebra with a non-standard test measure corresponding to the generating series of the L-values. A geometric interpretation of this non-standard basic function was conjectured by E. Frenkel and B.C. Ngô [FN11, Ngo14] and subsequently established by A. Bouthier, B.C. Ngô and Y. Sakellaridis [BNS16]. This framework aligns with the visionary proposal of A. Braverman and D. Kazhdan [BK99, BK00, BK02, BK03], who sought to generalize the Godement-Jacquet theory to arbitrary reductive groups. L. Lafforgue also has further expansions in [Laf14, Laf15, Laf16]. Significant progress has been made toward the Braverman-Kazhdan conjectures, particularly regarding the theory of gamma factors for finite reductive groups, as proved by S. Cheng and B.C. Ngô [CN18], T.H. Chen [Che22] and G. Laumon and E. Letellier [LL23]. Furthermore, the search for a generalized non-linear Fourier transform and its associated Poisson summation formula has seen recent advancements in the work of Z. Luo and B.C. Ngô [Luo19, LN24], Y. Choie, J.R. Getz, C.H. Hsu, B. Liu and S. Leslie [GL19, Get20, GL21, Get25, CG25, GHL25], G. Laumon and E. Letellier [LL24], and in a series of joint studies by D. Jiang, Z. Luo, G. Xi, L. Zhang and the author [JL22, JL23, JLZ24, JL25, JLX26].

1.1 Relative Functoriality Conjecture

Through the influential work of J. Guo, H. Jacquet, D. Jiang, K.F. Lai, E.Z. Lapid, Z. Mao, S. Rallis, and Y. Ye [Guo96, JL85, Jac86, Jac87, Ye89, JR92, JY92, Jac92, JMR99, MR99, JLR04, Lap06] on the relative trace formula and others, it has become increasingly evident that the functoriality conjecture admits a natural generalization to a broader class of G-spaces known as spherical varieties, which include symmetric spaces as a prominent subclass. Notably, the group case can be interpreted as a special instance of a symmetric space, wherein the group $\mathrm{G} \times \mathrm{G}$ acts on G via left and right multiplication. Spherical varieties are expected to possess associated L-groups, and according to [SV17], any morphism of L-groups ${}^{L}\mathrm{G}_{\mathrm{X}_1} \to {}^{L}\mathrm{G}_{\mathrm{X}_2}$ between the L-groups of two spherical varieties X_1 and X_2 should induce relations between their automorphic spectra.

In this context, the relative trace formula is envisioned as a distribution on the quotient stack $\mathcal{X} := [\mathrm{X} \times \mathrm{X}/\mathrm{G}]$, where X is a G-space, equipped with both geometric and spectral expansions, see [Sak16]. The overarching goal is to deduce relations between the spectral sides of two such relative trace formulas by comparing their corresponding geometric sides.

In [Sak21], Y. Sakellaridis proposed a new form of comparison for spherical varieties of rank one, that is, in cases where the associated L-groups satisfy ${}^{L}\mathrm{G}_{\mathrm{X}} = \mathrm{SL}_2$ or PGL_2. The central idea is to compare the relative trace formula associated with an arbitrary rank one spherical variety with the corresponding Kuznetsov trace formula, which is conjectured to represent a fundamental case in this broader framework of comparisons. As far as current understanding goes, there are principally two reasons for favoring the Kuznetsov trace formula over the classical Arthur-Selberg trace formula in this context. First, given a spherical G-variety X, the spectral side of the Kuznetsov trace formula is weighted by

$$\frac{1}{L(\varphi, \mathrm{Ad}, 1)}$$

while the relative trace formula will have an additional factor

$$\frac{L_{\mathrm{X}}(\varphi)}{L(\varphi, \mathrm{Ad}, 1)},$$

where φ is a global Arthur parameter. In the Arthur-Selberg trace formula case, we have $L_{\mathrm{X}}(\varphi) = L(\varphi, \mathrm{Ad}, 1)$. The second reason arises from geometric considerations, which lie somewhat beyond the scope of this thesis; we refer the interested reader to [Sak23b] for further details.

We now outline the strategy for approaching the functoriality conjecture for rank-one spherical varieties via trace formulas. Let X be a rank one spherical variety appearing in Table 2.5, ${}^L\mathrm{G}_{\mathrm{X}} = \mathrm{SL}_2$ or PGL_2, and write $\mathrm{G}_{\mathrm{X}} = \mathrm{PGL}_2$ or SL_2 respectively.

Locally, let $\mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G})$ be the push-forward of Schwartz measures $\mathcal{S}(\mathrm{X} \times \mathrm{X})$ to $\mathrm{X} \times \mathrm{X} /\!\!/ \mathrm{G} \cong \mathbb{A}^1$ along certain canonical map as in Theorem 2.3.1. Let $\mathbb{L}_{\mathrm{X}}^{\circ}$ be the push-forward image of the probability measure, which is equal to the characteristic function on $\mathrm{X}(\mathfrak{o}_{\mathrm{F}}) \times \mathrm{X}(\mathfrak{o}_{\mathrm{F}})$ times an invariant measure. On the Kuznetsov trace formula side, let $\mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}_{\psi} \backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi)$ be an enlarged space of test measures, and $f_{L_{\mathrm{X}}}$ be the non-standard basic vector determined by L_{X}, that we will explain in Section 2.2. Due to [Sak21], there is a linear isomorphism

$$\mathcal{T} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}_{\psi} \backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \to \mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G}),$$

with an explicit formula given by a composition of classical Fourier transforms with certain simple correction factors. In the first part of the thesis, we establish the fundamental lemma for rank one spherical varieties of classical type. Specifically, we demonstrate that the inverse of the transfer operator $\mathcal{T}$ maps $\mathbb{L}_{\mathrm{X}}^{\circ}$ to an explicit multiple of $f_{L_{\mathrm{X}}}$. Furthermore, we show that this mapping is equivariant with respect to the action of Hecke algebras, thereby extending the matching from the unit element to the full Hecke algebra.

Globally, the relative trace formula should be viewed as a distribution

$$\mathrm{RTF}_{\mathfrak{X}} : \mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G}) \to \mathbb{C}.$$

The geometric expansion is given as the evaluation map in [Sak16]. We still need the spectral expansion. On the Kuznetsov trace formula side, we have to extend the Kuznetsov trace formula to these enlarged test measures

$$\mathrm{KTF} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}_{\psi} \backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \to \mathbb{C}.$$

The Kuznetsov trace formula with non-standard test measures may exhibit divergence. However, a proposed method involves deforming the test measures with a complex number s to ensure convergence for sufficiently large real parts of s, i.e., $\mathrm{Re}(s) \gg 0$, following the approach of in [Sak19]. In this framework, the functional equation of the pertinent L-function is encoded by a Hankel transform

$$\mathcal{H} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}, \psi\backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \to \mathcal{S}^{-}_{L_{\mathrm{X}^\vee}}(\mathrm{N}, \psi\backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi). \tag{1.1}$$

This process is expected to yield a Poisson summation formula, thereby providing the meromorphic continuation of the Kuznetsov trace formula. Consequently, these functional equations facilitate control over the Kuznetsov trace formula within its region of divergence, enabling the application of the Phragmen-Lindelof principle to its full spectral decomposition. Building upon this methodology, we provide the complete derivation for the extension of the Kuznetsov trace formula corresponding to the standard L-functions of GL_2 in Section 3.4.

Subsequently, local transfer operators are employed to facilitate the formulation of another Poisson summation formula, which is instrumental in comparing the geometric sides of the relative trace formulas.

$$\begin{tikzcd}
\mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}_\psi\backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \arrow[rr, "\mathcal{T}"] \arrow[dr, "\mathrm{KTF}"'] & & \mathcal{S}(\mathrm{X}\times\mathrm{X}/\mathrm{G}), \arrow[dl, "\mathrm{RTF}_{\mathcal{X}}"] \\
& \mathbb{C} &
\end{tikzcd}$$

see [Sak19, Section 4].

For each X-distinguished cuspidal representation π of G, let $\mathrm{RTF}(\pi)$ be the π-component in the spectral expansion. For every cuspidal representation σ of G_{X}, let $\mathrm{KTF}(\sigma)$ be the σ-component in the spectral expansion of KTF. Let π_0 be an X-distinguished cuspidal representation of G, with the fundamental lemma for the full Hecke algebra, in principle, we will be led to the following identity

$$\sum_{\pi} \mathrm{RTF}(\pi) = \sum_{\sigma} \mathrm{KTF}(\sigma),$$

where the summation on the left-hand-side is over X-distinguished cuspidal representations that are nearly isomorphic to π_0, that is, $\pi_\nu \cong \pi_{0,\nu}$ for all but finitely many places, and the summation on the right-hand-side is over all the cuspidal representations σ such that the Satake parameter of σ_ν is mapped to π_ν at each unramified place under the canonical map

$$\mathrm{G}_{\mathrm{X}}^{\vee} \to \mathrm{G}^{\vee}.$$

To achieve the functoriality described above, a cornerstone of the Beyond Endoscopy program, it is evident that L-functions also play a fundamental role. Within this framework, consistent with the proposal of R. Langlands, L-functions should be explicitly introduced to weight the trace formula, and their residues at poles are then used to detect the image of functoriality lifts. It is worth noting that the original idea involved weighting by the logarithmic derivatives of L-functions, a method now recognized as particularly challenging to implement in this context. Having L-functions involved in the trace formula, as pointed out in [Sak23a], functoriality and functional equation of L-functions should be studied hand-in-hand.

1.2 Braverman-Kazhdan-Ngo Proposal

A closely related objective is the derivation of functional equations for L-functions through the interplay of the Hankel transform and the Poisson summation formula. The classical approach, originating with B. Riemann's representation of the Riemann zeta function as the Mellin transform of a theta series

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty y^{\frac{s}{2}} \sum_{n=1}^{\infty} e^{-n^2\pi y}\, \mathrm{d}^\times y$$

to prove its functional equation using the Poisson summation formula for the theta series, remains a guiding principle. See [Rie59]. This idea was further developed by K. Iwasawa and J. Tate and was reformulated in Tate's thesis [Tat67] in the language of adeles using the theory of local and global harmonic analysis. R. Godement and H. Jacquet generalized this work to study the analytic properties of standard automorphic L-functions of GL_n. The key input from the harmonic analysis to Godement-Jacquet theory is the classical Fourier transform on the affine space Mat_n of $n \times n$ matrices and the global Poisson summation formula that is responsible for the functional equation of standard L-functions.

Around the year 2000, A. Braverman and D. Kazhdan ([BK99, BK00, BK02, BK03]) proposed a framework to establish the Langlands conjecture for general L-functions

$L(s,\pi,\rho)$ via local and global harmonic analysis on the group, which generalizes the method of Tate's thesis and the work of Godement and Jacquet to the great generality, which is refined by B. Ngo in [Ngo20]. We also refer the reader to [Luo24] and [JLX26]. Under this framework, if generalized Fourier transforms and Poisson summation formulae could be established, there is a hope to prove the Langlands conjecture on the meromorphic continuation and the global functional equation of automorphic L-functions. The difficult part also lies in establishing the Poisson summation formula. We also refer to the work of D. Jiang and Z. Luo for establishing a version of Poisson summation formulae on $\mathbb{G}_m$ with automorphic representations involved; see [JL23].

Motivated by the framework of [FLN10], one may consider the use of non-standard test functions, which should be fundamentally related to the Braverman-Kazhdan-Ngô proposal, within the trace formula. This choice naturally leads to the appearance of L-functions in the spectral expansion. By carefully analyzing the geometric side, one could potentially extract information about L-functions. A key ingredient in this approach is a Poisson summation formula on the Hitchin base. As we discussed above, again, the Kuznetsov trace formula appears to serve as the foundation for all relative trace formulas. In the second part of the thesis, we introduce Godement-Jacquet test functions into the Kuznetsov trace formula and establish a global Poisson summation formula corresponding to the local Hankel transforms computed by Jacquet in [Jac03] for the case of GL_2. Specifically, we prove the commutativity of the following diagram:

$$\begin{tikzcd}[column sep=small]
\mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi) \arrow[rr, "\mathcal{H}_{\mathrm{Std},\mathbb{A}}"] \arrow[dr, "\mathrm{KTF}"] & & \mathcal{D}^-_{L\left(\mathrm{Std}^\vee,\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi) \arrow[dl, "\mathrm{KTF}"] \\
& \mathbb{C} &
\end{tikzcd}$$

to obtain the analytic continuation and functional equations of standard L-functions of GL_2. In the above diagram,

$$\mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash\mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi) := \otimes'_\nu \mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}_\nu,\psi_\nu\backslash\mathrm{G}_\nu/\mathrm{N}_\nu,\psi_\nu),$$

the restricted tesor product of local spaces with basic vectors specified, and the local space

$$\mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}_\nu,\psi_\nu\backslash\mathrm{G}_\nu/\mathrm{N}_\nu,\psi_\nu)$$

is defined to be the space of orbital integrals of Schwartz functions on $\mathrm{Mat}_2(k_\nu)$ with a suitable normalization. Using the coordintes of the maximal torus, KTF has the following explicit expression:

$$\mathrm{KTF}_{L(\mathrm{Std},\frac{1}{2})}(f) := \sum_{t\in \mathrm{T}(k)} \frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t) + \sum_{t_1\in k^\times} \mathcal{O}_{u,1,\mathbb{A}}(f)(t_1) + \sum_{t_2\in k^\times} \mathcal{O}_{u,2,\mathbb{A}}(f)(t_2),$$

a summation over the rational points of the maximal torus of GL_2 with boundary terms explicitly given, which are hidden in the expression of local Hankel transforms. In tandem with the Poisson summation formula, we derive the full spectral expansion of the Kuznetsov trace formula for the space of non-standard test measures related to the standard L-functions of GL_2, which provides a compelling validation of the proposed methodology for the global comparison of rank one spherical varieties, and a self-contained proof of the functional equation for the standard L-functions of GL_2.

1.3 Organization of the Thesis

Chapter 2 begins with an introduction to the background in Section 2.1, followed by Section 2.2, which focuses on the non-standard test measures for the Kuznetsov trace formula on SL_2 and PGL_2.

In Section 2.3.1, we review the transfer operator constructed in [Sak21], and Section 2.3.2 describes the construction of transfer operators via Weil representations, following [GS12]. These operators are closely related to those studied in Section 2.3.1. Once the relationship between the transfer operators arising from Weil representations and those defined in [Sak21] is established, the fundamental lemma for the unit elements follows from an explicit computation. Furthermore, the extension to the full Hecke algebra is obtained by demonstrating the compatibility of the Hecke actions, as predicated on the methods of S. Rallis [Ral82].

The ensuing sections, Section 2.4, Section 2.5, Section 2.6, and Section 2.7, are devoted to the case-by-case analysis required to establish the fundamental lemma across all classical types.

As for the second part of the thesis, Chapter 3 starts with an introduction to the Braverman-Kazhdan-Ngô proposal. Section 3.2 begins with a review of local conjectures of Braverman-Kazhdan-Ngô, and specializes to the case of GL_2 in Section 3.2.1. In this section, we will study properties of local orbital integrals and recall the Hankel transforms established by Jacquet in [Jac03]. Section 3.3 is about the global Poisson summation formula conjecture. In this section, we will write down the global Poisson summation formula based on the local Hankel transforms and give two proofs of it for the standard L-functions of GL_2. In the following Section 3.4, we will establish the Kuznetsov trace formula with Schwartz functions from 2×2 matrices. This can be viewed as one example of a trace formula with non-standard test functions. It turns out that the Poisson summation formula that we constructed in the previous section is closely related to the geometric side of the Kuznetsov trace formula. Then, in Section 3.4.3, we will study the analytic continuation of the geometric side and the Eisenstein series part from the spectrum side of the trace formula. For the geometric side, we will use the Poisson summation formula in Section 3.3, and the Eisenstein series part is essentially reduced to Tate's thesis. In this way, we can establish the analytic continuation of the cuspidal part. The next Section 3.4.4 will focus on the functional equation. It also consists of two parts, the first of which is the functional equation of the geometric part corresponding to the Poisson summation formula. Then, we will compare the Eisenstein series parts according to the functional equation of the GL_1 integrals and obtain the functional equation for the cuspidal part. Finally, in the last Section 3.4.5, we will isolate the spectrum and obtain the analytic continuation and functional equation of standard L-functions for one single cuspidal representation.

1.4 Notations

- For a finite set S, we will use $\#\mathrm{S}$ to denote the cardinality of S, and 1_{S} the characteristic function of S.

- Let k be a global field and F be a non-Archimedean local field of characteristic zero, $\mathfrak{o}_{\mathrm{F}}$ its ring of integers, and $\mathfrak{p}_{\mathrm{F}}$ its maximal ideal. Let q be the cardinality of the residue field $\kappa_{\mathrm{F}} := \mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}$. Assume q is odd. Fix a uniformizer ϖ_{F} and for any $x \in \mathrm{F}$, denote
$$\mathrm{ac}(x) := \frac{x}{\varpi_{\mathrm{F}}^{\mathrm{val}(x)}} \in \mathfrak{o}_{\mathrm{F}}^{\times},$$
where $\mathrm{val} : \mathrm{F}^{\times} \to \mathbb{Z}$ is the normalized valuation map such that $\mathrm{val}(\varpi_{\mathrm{F}}) = 1$. Fix a non-trivial unramified character ψ of F.

- Let
$$\zeta_{\mathrm{F}}(s) := \frac{1}{1-q^{-s}}$$
be the local zeta function, where $s \in \mathbb{C}$.

- Let $\mathbb{D}$ be an algebra over F of one of the following types:

  [(1)] the field F itself;

  [(2)] the split etale quadratic extension of F, i.e., $\mathbb{D} = \mathrm{F} \times \mathrm{F}$;

  [(3)] the matrix algebra $\mathrm{Mat}_2(\mathrm{F})$ of 2×2 matrices over F. In all the cases, we identify F with the subfield of $\mathbb{D}$ consisting of scalar multiples of the identity. Equip $\mathbb{D}$ with the F-involution $\bar{\cdot}$, together with the (reduced) trace and (reduced) norm $\tau, \nu : \mathbb{D} \to \mathrm{F}$ as follows:

  [(1)] If $\mathbb{D} = \mathrm{F}$, let $\overline{x} := x$, $\tau(x) := x$, and $\nu(x) := x$.

  [(2)] If $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, and $x = (a, b)$, let $\overline{x} := (b, a)$, $\tau(x) := a + b$ and $\nu(x) := ab$.

  [(3)] If $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$, and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\overline{x} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\tau(x) := a + b$ and $\nu(x) := ad - bc$.

- When the meaning is clear from the context, for a variety X over F, we denote the set of F-rational points X(F) simply by X.

- For a smooth variety X over F, we denote by $\mathcal{F}(\mathrm{X})$ the space of Schwartz functions on the rational points of X. It means smooth functions of compact support. Similarly, we will use $\mathcal{S}(\mathrm{X})$ to denote the space of Schwartz measures and $\mathcal{D}(\mathrm{X})$ for the space of Schwartz half-densities. Often X(F) is a homogeneous G(F)-space for some reductive group G, and admits an invariant positive measure $\mathrm{d}x$, then we have $\mathcal{S}(\mathrm{X}) = \mathcal{F}(\mathrm{X}) \cdot \mathrm{d}x$ and $\mathcal{D}(\mathrm{X}) = \mathcal{F}(\mathrm{X}) \cdot (\mathrm{d}x)^{\frac{1}{2}}$.

- Unless otherwise specified, if X is a homogeneous G-variety that admits an invariant rationally defined non-vanishing top degree differential form, we use $\mathrm{d}x$ to denote the induced local unramified measure, which satisfies
$$\tau(\mathrm{X}) := \int_{\mathrm{X}(\mathfrak{o}_{\mathrm{F}})} 1 \, \mathrm{d}x = \frac{\#\mathrm{X}(\kappa_{\mathrm{F}})}{q^{\dim \mathrm{X}}},$$

where # means the cardinality. See [Wei82, Theorem 2.2.5]. Note that in the case that $\mathrm{X} = \mathbb{G}_a$, $\mathrm{d}x$ is also the self-dual Haar measure on F. We also equip $\mathbb{D}$ with the product Haar measure. It might be a little bit confusing between $\mathrm{X} = \mathbb{G}_m$ and $\mathrm{X} = \mathbb{G}_a$. When we use F or $\mathrm{F}^\times$ (instead of $\mathbb{G}_a$ and $\mathbb{G}_m$) to denote the integral domain, $\mathrm{d}x$ is always reserved to denote the additive Haar measure, i.e., for a function f on $\mathbb{G}_m$, we will write

$$\int_{\mathbb{G}_m} f(x)\,\mathrm{d}x = \int_{\mathrm{F}^\times} f(x)\frac{\mathrm{d}x}{|x|}$$

interchangeably. To emphasize the choice of the multiplicative Haar measure on the torus, we adopt the notation $\mathrm{d}^\times x$ where appropriate

- For the notation of the Fourier transforms, let f be a measure on $\mathbb{G}_a$, for $s \in \mathbb{C}$, we use $(|\cdot|^s\psi(\cdot))\star$ to denote the operator of multiplicative convolution by the measure $|x|^s\psi(x)\,\mathrm{d}x$:

  $$(|\cdot|^s\psi(\cdot))\star f(y) := \int_{\mathrm{F}} |x|^s\psi(x)f(x^{-1}y)\,\mathrm{d}x.$$

  More generally, let f be an half-density on T and $\lambda^\vee$ be any cocharacter of T, $s \in \mathbb{C}$, we define

  $$\mathcal{F}^{\psi}_{\lambda^\vee,s}f(\xi) := \int_{\mathbb{G}_m} |x|^s\psi(x)f(\lambda^\vee(x)^{-1}\xi)\,\mathrm{d}^\times x,$$

  whenever this integral makes sense. If the integral does not converge, then the convolution should be understood as the Fourier transform of measures or distributions.

- If ϕ is a function on F, we also use the usual Fourier transform on the Affine line

  $$\mathbb{F}_\psi(\phi)(x) := \int_{\mathrm{F}} \phi(y)\psi^{-1}(xy)\,\mathrm{d}y.$$

  Then if f is represented by some function $\phi(x)\,\mathrm{d}x$, we have

  $$(|\cdot|^s\psi(\cdot))\star f(y) = (|\cdot|^s \circ \mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot|^{1-s})(\phi)(y)\,\mathrm{d}y,$$

  where $\iota(\phi)(x) := \phi(x^{-1})$. When ϕ is a multi-variable function, we will use $\mathbb{F}_{\psi,i}(\phi)$ to denote this Fourier transform for the i-th variable with other variables fixed.

We will also identify $\mathrm{Mat}_2 \cong \mathbb{A}^4 : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto (x_{11}, x_{12}, x_{21}, x_{22})$ and use $\mathbb{F}_{\psi,i}$ to denote the Fourier transform with respect to the i-th coordinate using this identification.

Chapter 2

Relative Functoriality

2.1 Introduction

We begin with an overview of the relative trace formula. Let k be a global field, $\mathbb{A}$ its ring of adeles, G a split reductive group over k, and X a smooth affine spherical variety over k. For the purpose of this paper, we may take

$$\mathcal{F}(\mathrm{X}(\mathbb{A})) := \otimes'_{\nu}\mathcal{F}(\mathrm{X}(k_{\nu})),$$

where the tensor product is over all the places of k with respect to the characteristic function $1_{\mathrm{X}(\mathfrak{o}_{\mathrm{F}})}$ of $\mathrm{X}(\mathfrak{o}_{\nu})$. The X-theta series is the following functional on $\mathcal{F}(\mathrm{X}(\mathbb{A}))$:

$$\Phi \mapsto \theta(\Phi, g) := \sum_{\gamma \in \mathrm{X}(k)} \Phi(\gamma g) \in \mathcal{C}^{\infty}([\mathrm{G}]),$$

where the sum is absolutely convergent since $\mathrm{X}(k)$ is discrete in $\mathrm{X}(\mathbb{A})$ as X is affine.

Roughly speaking, the relative trace formula is the Plancherel formula for $\mathcal{L}^2([\mathrm{G}])$, applied to the inner product of two theta series:

$$\mathrm{RTF}_{\mathfrak{X}} : \mathcal{F}(\mathrm{X}(\mathbb{A})) \otimes \mathcal{F}(\mathrm{X}(\mathbb{A})) \ni \Phi_1 \otimes \Phi_2 \mapsto \int_{[\mathrm{G}]}^{*} \theta(\Phi_1, g)\theta(\Phi_2, g)\,\mathrm{d}g.$$

This inner product does not converge in general and needs to be regularized. In the case that X is smooth affine and G is split, there is a way to regularize it as described in [Sak16].

As for the Kuznetesov trace formula, one can define KTF similarly, but with non-standard spaces of test measures, which we will describe in Section 2.2.

For the purpose of this thesis, let us consider the case related to rank one spherical varieties. The list of spherical varieties of rank 1 consists of a finite number of families, classified by Ahiezer [Ahi83], also see [KVS06] and [Was96]. Up to an action of the center, the affine homogeneous spherical varieties over an algebraically closed field in characteristic 0 are listed in Table 2.5.

For each spherical variety, L_{X} stands for a $\frac{1}{2}\mathbb{Z}$-graded representation $\rho = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} \rho_n$ of its dual group $\mathrm{G}_{\mathrm{X}}^{\vee}$, which is called the L-value asssociated to X, thinking of the L-value

$$\prod_n L(\rho_n \circ \varphi, n)$$

attached to any Langlands parameter φ into $\mathrm{G}_{\mathrm{X}}^{\vee}$. And the L-value here is determined in terms of the geometry of X. See also Table 2.5. For each L_{X}, we define a space of non-standard test measures in Section 2.2, together with basic vectors

$$f_{L_{\mathrm{X}}} := \begin{cases} f_{L(\mathrm{Std},\mathrm{s}_1)L(\mathrm{Std},\mathrm{s}_2)} & \text{for} \quad \mathrm{G}_{\mathrm{X}}^{\vee} = \mathrm{SL}_2; \\ f_{L(\mathrm{Ad},s_0)} & \text{for} \quad \mathrm{G}_{\mathrm{X}}^{\vee} = \mathrm{PGL}_2. \end{cases}$$

Here s_1, s_2 and s_0 are also listed as in the same table, depending on X. We take

$$\mathcal{F}_{L_{\mathrm{X}}}(\mathrm{N}(k_\nu), \psi_\nu \backslash \mathrm{G}_{\mathrm{X}}(k_\nu))$$

to be the Schwartz sections of certain bundle over $\overline{\mathrm{N}\backslash\mathrm{G}_{\mathrm{X}}}$, together with basic vector

$$\Phi_{L_{\mathrm{X}}} := \begin{cases} \Phi_{L(\mathrm{Std},\mathrm{s}_1)L(\mathrm{Std},\mathrm{s}_2)} & \text{for} \quad \mathrm{G}_{\mathrm{X}}^{\vee} = \mathrm{SL}_2; \\ \Phi_{L(\mathrm{Ad},s_0)} & \text{for} \quad \mathrm{G}_{\mathrm{X}}^{\vee} = \mathrm{PGL}_2. \end{cases}$$

Here again $\Phi_{L_{\mathrm{X}}}$ is defined in Section 2.2. Then we have the Poincaré series

$$\mathcal{P}(\Phi, g) := \sum_{\gamma \in \mathrm{N}(k)\backslash \mathrm{G}_{\mathrm{X}}(k)} \Phi(\gamma g)$$

for $\Phi \in \mathcal{F}_{\mathrm{X}}(\mathrm{N}(\mathbb{A}), \psi\backslash\mathrm{G}_{\mathrm{X}}(\mathbb{A})) := \otimes'_\nu \mathcal{F}_{\mathrm{X}}(\mathrm{N}(k_\nu), \psi_\nu\backslash\mathrm{G}_{\mathrm{X}}(k_\nu))$. Let $\mathcal{F}(\mathrm{N}(\mathbb{A}), \psi\backslash\mathrm{G}(\mathbb{A}))$ be the space of standard test functions, then the Kuznetsov trace formula is the decomposition of

$$\mathrm{KTF} : \mathcal{F}_{\mathrm{X}}(\mathrm{N}(\mathbb{A}), \psi\backslash\mathrm{G}_{\mathrm{X}}(\mathbb{A})) \otimes \mathcal{F}(\mathrm{N}(\mathbb{A}), \psi\backslash\mathrm{G}_{\mathrm{X}}(\mathbb{A})) \ni \Phi_1 \otimes \Phi_2 \mapsto \int^*_{[\mathrm{G}_{\mathrm{X}}]} \mathcal{P}(\Phi_1, g)\mathcal{P}(\Phi_2, g)\,\mathrm{d}g.$$

Of course, the integral here also needs to be regularized.

The ultimate objective of this framework is to establish a rigorous global comparison between $\mathrm{RTF}_{\mathfrak{X}}$ and KTF. Locally, the smooth transfer has already been established in [Sak21]. Let $\pi : \mathrm{X} \times \mathrm{X} \to \mathrm{X} \times \mathrm{X} /\!\!/ \mathrm{G}$ be the canonical quotient map to the affine, invariant-theoretic quotient Spec $\mathrm{F}[\mathrm{X} \times \mathrm{X}]^{\mathrm{G}}$, and the image $\pi_* \mathcal{S}(\mathrm{X} \times \mathrm{X})$ of the push-forward measure will be denoted by $\mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G})$. Y. Sakellaridis [Sak21] proved there is a linear isomorphism

$$\mathcal{T} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}_{\psi} \backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \to \mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G}),$$

with an explicit formula given by a composition of classical Fourier transforms with certain simple correction factors. For more details, see Theorem 2.3.1 in Section 2.3.1.

To achieve the relative Langlands functoriality via comparing the relative trace formula for X and the Kuznetsov trace formula, we also need the appropriate fundamental lemma, and the first main result of this paper is the following fundamental lemma, which is a combination of Theorem 2.4.3, Theorem 2.5.3, Theorem 2.6.3, and Theorem 2.7.9:

Theorem 2.1.1. *Let* X *be a rank one spherical variety and* $\mathfrak{C}_{\mathrm{X}} = (\mathrm{X} \times \mathrm{X}) /\!\!/ \mathrm{G}$. *Fix the isomorphism* $\mathfrak{C}_{\mathrm{X}} \cong \mathbb{A}^1$ *as in Theorem 2.3.1, and under the transfer operator*

$$\mathcal{T} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}, \psi \backslash \mathrm{G}_{\mathrm{X}}/\mathrm{N}, \psi) \to \mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G})$$

there,

$$\mathcal{T}^{-1}(\mathbb{L}^{\circ}_{\mathrm{X}}) = \frac{q^{\dim \mathrm{X}}}{\#\mathrm{X}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})} f_{L_{\mathrm{X}}}.$$

where

$$s_{\mathrm{X}} = \begin{cases} s_1 + s_2 = \frac{\dim \mathrm{X}}{2} & \textit{for} \quad \mathrm{G}^{\vee}_{\mathrm{X}} = \mathrm{SL}_2; \\ 2s_0 = \dim \mathrm{X} - 1 & \textit{for} \quad \mathrm{G}^{\vee}_{\mathrm{X}} = \mathrm{PGL}_2. \end{cases}$$

Here $\mathbb{L}^{\circ}_{\mathrm{X}}$ *is the push-forward image of the probability measure, which is equal to the characteristic function on* $\mathrm{X}(\mathfrak{o}_{\mathrm{F}}) \times \mathrm{X}(\mathfrak{o}_{\mathrm{F}})$ *times an invariant measure.*

Conjecture 2.1.2. *Theorem 2.1.1 holds for all rank one spherical varieties.*

Remark 2.1.3. *If we write* $\tau(\mathrm{X})$ *for the local Tamagawa measure of* X, *then we have*

$$\tau(\mathrm{SL}_2) = \tau(\mathrm{PGL}_2) = 1 - q^{-2} = \frac{1}{\zeta_{\mathrm{F}}(2)},$$

hence we can rewrite the above formula as

$$\mathcal{T}^{-1}(\mathbb{L}_{\mathrm{X}}^{\circ}) = \frac{\tau(\mathrm{G_X})}{\tau(\mathrm{X})\zeta_{\mathrm{F}}(s_{\mathrm{X}})} f_{L_{\mathrm{X}}}.$$

However, to globally implement the relative trace formula for establishing relative functoriality and the correlation between X-periods of automorphic forms and L-values, the fundamental lemma for the full Hecke algebra is required to effectively isolate the spectral contributions.

Fix a Borel subgroup $\mathrm{B} \subset \mathrm{G}$, the maximal torus $\mathrm{A} \subset \mathrm{B}$. Fix a hyperspecial vertex in the Bruhat-Tits building of G that lies in the apartment determined by A. It determines a hyperspecial maximal open compact subgroup K of G. Let $\mathrm{X}^{\circ} \subset \mathrm{X}$ be its open orbit under the B-action. Let $\mathrm{P(X)} := \{g \in \mathrm{G} \mid \mathrm{X}^{\circ} g = \mathrm{X}^{\circ}\}$. Let N be the unipotent radical of B, then the quotient $\mathrm{X}^{\circ} /\!\!/ \mathrm{N}$ is a homogeneous space under the action of A, whose action factors through the faithful action of a quotient $\mathrm{A} \twoheadrightarrow \mathrm{A_X}$, the universal Cartan of X. It is known that $\mathrm{A_X}$ is a quotient of P(X) through the Levi L(X):

$$\mathrm{P(X)} \to \mathrm{L(X)} \to \mathrm{A_X}.$$

This gives rise to a map with finite kernel between the complex dual tori

$$\mathrm{A}_{\mathrm{X}}^{\vee} \to \mathrm{A}^{\vee}. \tag{2.1}$$

While this map facilitates the study of the unramified spectrum of X, it necessitates a modification as formulated in [Sak13b]. Let $\delta_{(\mathrm{X})}^{\frac{1}{2}}$ be the square root of the modular character, which is defined as the quotient of the right by the left Haar measure of the group $\mathrm{B} \cap \mathrm{L(X)}$, considered as an unramified character of B and hence as an element of $\mathrm{T}^{\vee}$. It is stable under the little Weyl group $\mathbb{W}_{\mathrm{X}}$ of X, and we consider the normalized morphism

$$\mathrm{A}_{\mathrm{X}}^{\vee} \to \mathrm{A}^{\vee} : \widetilde{\chi} \mapsto \chi\delta_{(\mathrm{X})}^{\frac{1}{2}}, \tag{2.2}$$

where χ is the image of $\widetilde{\chi}$ under (2.1). Consider the spherical Hecke algebra $\mathcal{H}(\mathrm{G},\mathrm{K})$ of G. By restriction via the above normalized morphism between tori, we get a morphism of algebras

$$\lambda_{\mathrm{X}} : \mathcal{H}(\mathrm{G},\mathrm{K}) \cong \mathbb{C}[\mathrm{A}^{\vee}]^{\mathbb{W}_{\mathrm{G}}} \to \mathbb{C}[\mathrm{A}_{\mathrm{X}}^{\vee}]^{\mathbb{W}_{\mathrm{X}}} \cong \mathcal{H}(\mathrm{G_X}, \mathrm{G_X}(\mathfrak{o}_{\mathrm{F}})).$$

Theorem 2.1.4. *Let* X *be a rank one spherical variety of classical type and* $\mathfrak{C}_{\mathrm{X}} = (\mathrm{X} \times \mathrm{X}) /\!\!/ \mathrm{G}$*. Fix the isomorphism* $\mathfrak{C}_{\mathrm{X}} \cong \mathbb{A}^1$ *as in Theorem 2.3.1, and under the transfer operator*

$$\mathcal{T} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}, \psi \backslash \mathrm{G}_{\mathrm{X}} / \mathrm{N}, \psi) \to \mathcal{S}(\mathrm{X} \times \mathrm{X} / \mathrm{G})$$

there,

$$\mathcal{T}^{-1}(f \star \mathbb{L}^{\circ}_{\mathrm{X}}) = \lambda_{\mathrm{X}}(f) \star \frac{q^{\dim \mathrm{X}}}{\#\mathrm{X}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})} f_{L_{\mathrm{X}}}.$$

Conjecture 2.1.5. *Theorem 2.1.4 for all rank one spherical varieties.*

Remark 2.1.6. *The author is grateful to Yi Shan for noting that Conjecture 2.1.2 for* B_3''*,* D_4''*, and* G_2 *follows from isomorphisms*

$$\mathrm{G}_2 \backslash \mathrm{Spin}_7 \cong \mathrm{Spin}_7 \backslash \mathrm{Spin}_8 \cong \mathrm{SO}_7 \backslash \mathrm{SO}_8, \text{ and } \mathrm{SL}_3 \backslash \mathrm{G}_2 \cong \mathrm{SO}_6 \backslash \mathrm{SO}_7,$$

effectively reducing them to the classical cases established herein. I would also like to thank the anonymous referee of a preprint related to this work for pointing out that Conjecture 2.1.5 for these cases follows directly from [Sak13b, Theorem 1.4.1], and then the only remaining case is of F_4*, which is discussed in a recent preprint of [LW25].*

Remark 2.1.7. *Before moving on, let us clarify the notations. First, note that the Hecke algebra does not act on the space of push-forward of measures. However, the Bernstein center does, where the action is on the first copy, and the Bernstein center for the component of the spectrum corresponding to the unramified principal series is isomorphic to the Hecke algebra under the natural map. Of course, there is a more directed description of this action. Let us follow the conversion used in [Sak13a]. Let* Φ_1 *be a Schwartz function on* X *that is fixed by* K *and* $h \in \mathcal{H}(\mathrm{G}, \mathrm{K})$*, consider the right convolution, which is the push-forward under the action map*

$$h \star \Phi(x) := \int_{\mathrm{G}} \Phi(xg^{-1}) h(g) \, \mathrm{d}g.$$

Then for $\Phi_1 \otimes \Phi_2$*, we interpret the action of* h *on its push-forward measure as the image of* $(h \star \Phi_1) \otimes \Phi_2$*. If one wants to consider the action on the second copy, the action of the Bernstein center is also the image of the push-forward of* $\Phi_1 \otimes (h^{\vee} \star \Phi_2)$*.*

When $\mathrm{X} = \mathrm{H} \backslash \mathrm{G}$*, under the various canonical identifications*

$$(\mathrm{H}\backslash\mathrm{G} \times \mathrm{H}\backslash\mathrm{G}) /\!\!/ \mathrm{G} = \mathrm{G} \mathbin{\backslash\!\!\backslash} (\mathrm{G}/\mathrm{H} \times \mathrm{G}/\mathrm{H}) = (\mathrm{H}\backslash\mathrm{G}) /\!\!/ \mathrm{H} = \mathrm{H} \mathbin{\backslash\!\!\backslash} (\mathrm{G}/\mathrm{H}),$$

One can check that the Hecke algebra action is realized either as the $\star$ action on $\mathrm{H}\backslash\mathrm{G}$ *or the following right action of* $f \in \mathcal{H}(\mathrm{G},\mathrm{K})$ *on* K*-invariant functions on* G/H *defined by*

$$h \star \Phi(x) := \int_{\mathrm{G}} h(g)\Phi(gx)\,\mathrm{d}x,\ \Phi \in \mathcal{F}(\mathrm{G}/\mathrm{H})^{\mathrm{K}}.$$

Remark 2.1.8. *In view of the subsequent analysis involving Weil representations, we adopt the left* G*-action on* G/H *for all cases. Consequently, to ensure consistency with (2.2), the modular character* $\delta_{(\mathrm{X})}$ *is replaced by its inverse,* $\delta_{(\mathrm{X})}^{-1}$*, for the homomorphism between Hecke algebras.*

2.2 Test measures for the Kuznetsov formula

Let G be a split reductive group over F with fixed Borel pairs (B,A). Let $\mathrm{G}^\vee$ be the dual group with the corresponding Borel pair $(\mathrm{B}^\vee,\mathrm{A}^\vee)$. Let $X^*(\mathrm{A})$ be the F-rational characters of A and $X_*(\mathrm{A})$ be the F-rational co-characters of A. Then $X^*(\mathrm{A}) = X_*(\mathrm{A}^\vee)$ and $X^*(\mathrm{A}^\vee) = X_*(\mathrm{A})$. We denote by $X^*(\mathrm{A})^+$ and $X_*(\mathrm{A})^+$, respectively, the dominant cones in $X^*(\mathrm{A})$ and $X_*(\mathrm{A})$, respectively, with the positivity being associated to the chosen Borel subgroup. We denote by Δ the set of simple roots and $\Delta^\vee$ the set of simple co-roots, respectively. In this way, we obtain the based root datum $(X^*(\mathrm{A}),\Delta,X_*(\mathrm{A}),\Delta^\vee)$ of G with respect to (B,A) and the based root datum $(X_*(\mathrm{A}),\Delta^\vee,X^*(\mathrm{A}),\Delta)$ of $\mathrm{G}^\vee$ with respect to $(\mathrm{B}^\vee,\mathrm{A}^\vee)$.

Fix a hyperspecial vertex in the Bruhat-Tits building of G that lies in the apartment determined by A. It determines a hyperspecial maximal open compact subgroup K of G. Note that $\mathrm{K}_{\mathrm{A}} := \mathrm{A}\cap\mathrm{K}$ is a hyperspecial subgroup of A. Let A^+ be the image of $X_*(\mathrm{A})^+$ under the map $\mu \mapsto \mu(\varpi_{\mathrm{F}})$, where $\varpi_{\mathrm{F}} \in \mathrm{F}$ is a fixed uniformizer. We have the Cartan decomposition

$$\mathrm{G} = \mathrm{K}\mathrm{A}^+\mathrm{K}.$$

Only here we take the Haar measure on G such that the volume of K is 1. The spherical Hecke algebra of $\mathcal{H}(\mathrm{G},\mathrm{K})$ is defined as the convolution algebra of compactly supported K-bi-invariant $\mathbb{C}$-valued functions on G. The same definition applies to A with respect to K_{A} and the Weyl group $\mathbb{W}_{\mathrm{G}}$ acts on $\mathcal{H}(\mathrm{A},\mathrm{K}_{\mathrm{A}})$.

The Satake isomorphism, as is defined in [Sat63] yields an isomorphism of algebras:

$$\mathcal{H}(\mathrm{G},\mathrm{K}) \to \mathcal{H}(\mathrm{A},\mathrm{K}_\mathrm{A})^{\mathbb{W}_\mathrm{G}}, \quad h \mapsto \left(t \mapsto \delta_\mathrm{B}^{\frac{1}{2}}(t)\int_\mathrm{N} h(tn)\,\mathrm{d}n\right), \tag{2.3}$$

where δ_B is the modular character on B and $\mathrm{d}n$ is the Haar measure on N such that the volume of $\mathrm{N}\cap\mathrm{K}$ is 1.

Note that $X_*(\mathrm{A}) \cong \mathrm{A}/\mathrm{K}_\mathrm{A} \colon \mu \mapsto \mu(\varpi_\mathrm{F})$, which implies that

$$\mathcal{H}(\mathrm{A},\mathrm{K}_\mathrm{A})^{\mathbb{W}_\mathrm{G}} \cong \mathbb{C}[X_*(\mathrm{A})]^{\mathbb{W}_\mathrm{G}} = \mathbb{C}[X^*(\mathrm{A}^\vee)]^{\mathbb{W}_\mathrm{G}}.$$

Write

$$\mathrm{Sat}: \mathcal{H}(\mathrm{G},\mathrm{K}) \to \mathcal{H}(\mathrm{A},\mathrm{K}_\mathrm{A})^{\mathbb{W}_\mathrm{G}} \to \mathbb{C}[X^*(\mathrm{A}^\vee)]^{\mathbb{W}_\mathrm{G}}.$$

It is known that for any finite-dimensional algebraic representation ρ of $\mathrm{G}^\vee(\mathbb{C})$, we must have $\mathrm{tr}\ \rho \in \mathbb{C}[X^*(\mathrm{A}^\vee)]^{\mathbb{W}_\mathrm{G}}$, and there exists a canonical one-to-one correspondence between the set $\Pi_\mathrm{F}(\mathrm{G},\mathrm{K})$ of equivalent classes of irreducible admissible spherical representations of G with respect to K and the set of conjugacy classes $\mathfrak{c}$ of $\mathrm{A}^\vee(\mathbb{C})/\mathbb{W}_\mathrm{G}$.

More precisely, let $\mathfrak{s} \in X^*(\mathrm{A}) \otimes_\mathbb{Z} \mathbb{C}/\mathbb{W}_\mathrm{G}.\left(\frac{2\pi i}{\log q}X^*(\mathrm{A})\right)$, where following [Sat63], we use

$$\mathbb{W}_\mathrm{G}.\left(\frac{2\pi i}{\log q}X^*(\mathrm{A})\right)$$

to denote the group of affine transformations of $X^*(\mathrm{A}) \otimes_\mathbb{Z} \mathbb{C}$ generated by the Weyl group $\mathbb{W}_\mathrm{G}$ and by the group of translations defined by $\frac{2\pi i}{\log q}X^*(\mathrm{A})$. On the one hand, denote $\chi(\mathfrak{s})$ to be the corresponding unramified character of A, and let $\mathrm{I}(\chi(\mathfrak{s}))$ be the normalized induced representation, whose underlying space consists of locally constant functions Φ on G with compact support module B, satisfying

$$\Phi(gb) = \delta_\mathrm{B}^{-\frac{1}{2}}(b)\chi(\mathfrak{s})(b)\Phi(g),\ \forall g \in \mathrm{G}, b \in \mathrm{B},$$

with the action of G given by $g.\Phi(x) := \Phi(g^{-1}x)$. $\mathrm{I}(\chi(\mathfrak{s}))$ admits a unique unramified subquotient, which we denote $\pi(\mathfrak{s})$.

On the other hand, view $\mathfrak{s} \in X_*(\mathrm{A}^\vee) \otimes_\mathbb{C} \mathbb{C}/\mathbb{W}_\mathrm{G}.\left(\frac{2\pi i}{\log q}X_*(\mathrm{A}^\vee)\right)$ and let $\mathfrak{c}(\mathfrak{s}) = q^\mathfrak{s} \in \mathrm{A}^\vee(\mathbb{C})/\mathbb{W}$ be the corresponding conjugacy class. Then we have

$$\mathrm{tr}_{\ \pi(\mathfrak{s})}(h) = \mathrm{Sat}(h)(\mathfrak{c}(\mathfrak{s}))$$

for any $h \in \mathcal{H}(\mathrm{G},\mathrm{K})$.

Example 2.2.1. *Let* $\mathrm{G} = \mathrm{SL}_2$, $\mathrm{B} := \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$ *and* $\mathrm{A} := \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$. *Let* $s \in X^*(\mathrm{A}) \otimes_{\mathbb{Z}} \mathbb{C} = X_*(\mathrm{A}^\vee) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}$, *which gives the unramified character* $\chi(s)$ *of* A

$$\mathrm{A} \to \mathbb{C}^\times : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto |t|^s.$$

and the normalized induced representation $\mathrm{I}(\chi(s))$. *Let* $\pi(s)$ *be the unique irreducible unramified subquotient of* $\mathrm{I}(\chi(s))$. *On the other hand,* $\mathfrak{c}(s)$ *is represented by* $\left(\begin{pmatrix} q^s & \\ & 1 \end{pmatrix}\right)$, *in the maximal torus of* PGL_2. *The Satake transform states that we have an isomorphism*

$$\mathrm{Sat} : \mathcal{H}(\mathrm{SL}_2, \mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})) \to \mathbb{C}[\mathrm{A}^\vee]^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{C}[x, x^{-1}]^{\mathbb{Z}/2\mathbb{Z}}$$

such that for any $h \in \mathcal{H}(\mathrm{SL}_2, \mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}}))$, *we have*

$$\mathrm{tr}_{\pi(s)}(h) = \mathrm{Sat}(h)(q^s).$$

Similarly for $\mathrm{G} = \mathrm{PGL}_2$.

Let ρ be an algebraic representation of $\mathrm{G}^\vee(\mathbb{C})$. Consider the L-value

$$L(\rho, s) := \frac{1}{\det(1 - q^{-s}\rho)} \in \mathbb{C}(\mathrm{G}^\vee)^{\mathrm{G}^\vee},$$

where $\mathbb{C}(\mathrm{G}^\vee)^{\mathrm{G}^\vee}$ is the space of invariant rational functions on $\mathrm{G}^\vee$ under the adjoint action. When $\rho = \bigoplus_{i=1}^{m} \rho_i$ is reducible, we can allow $s = (s_i)_{i=1}^m$ to be a collection of complex numbers, and define

$$L(\rho, s) := \prod_{i=1}^{m} L(\rho_i, s_i).$$

We proceed with a single s, and the arguments for general s are similar. The L-factor can be written as a formal Taylor series in q^{-s}:

$$L(\rho, s) = \sum_{n=0}^{\infty} q^{-ns} \mathrm{tr}\,(\mathrm{Sym}^n \rho).$$

The generating series of the local L-function is the Whittaker function

$$\Phi_{L(\rho,s)} := \sum_{n=0}^{\infty} q^{-ns} \mathrm{Sat}^{-1}(\mathrm{Sym}^n \rho) \star 1_{\mathrm{N}\backslash\mathrm{G}(\mathfrak{o}_{\mathrm{F}})}.$$

When $\mathrm{G} = \mathrm{PGL}_2$ or SL_2, let B be the Borel subgroup consisting of upper triangular matrices, A the maximal torus consisting of diagonal matrices. Let N be the unipotent radical of B consisting of strictly upper triangular matrices. Choose $\mathbf{w} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to be the non-trivial Weyl element. Let $\mathfrak{C}$ be the quotient $\mathrm{N}\backslash\mathrm{G} /\!\!/ \mathrm{N}$, and $\mathfrak{C}^\circ$ its open subset corresponding to the open Bruhat cell. We embed A into it by $t \mapsto [\mathbf{w}t]$, the class of the element $\mathbf{w}t$. Let

$$\mathcal{S}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$$

be the space of measures on A of the form

$$f(t) = \frac{\delta_{\mathrm{B}}(t)}{1-q^{-2}} \int_{\mathrm{N}\times\mathrm{N}} \Phi(n_1\mathbf{w}tn_2)\psi^{-1}(n_1n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \cdot \,\mathrm{d}t, \tag{2.4}$$

where recall $\mathrm{d}n$ is such that $\mathrm{vol}(\mathrm{N}(\mathfrak{o}_{\mathrm{F}}), \mathrm{d}n) = 1$, $\mathrm{d}t$ is such that $\mathrm{vol}(\mathrm{A}(\mathfrak{o}_{\mathrm{F}}), \mathrm{d}t) = 1-q^{-1}$, and Φ is a Schwartz function on G. We also fix the coordinates of A by

$$\xi(t) = e^{\alpha}(t), \text{ if } \mathrm{G} = \mathrm{PGL}_2,$$

and

$$\zeta(t) = e^{\frac{\alpha}{2}}(t), \text{ if } \mathrm{G} = \mathrm{SL}_2,$$

where α is the positive root, and we use the exponential notation to denote the corresponding character. Then $\mathfrak{C}^\circ$ can be identified with $\mathbb{G}_m$ or $\mathbb{G}_m^2$ with sections

$$\xi \mapsto \begin{pmatrix} 0 & -1 \\ \xi & 0 \end{pmatrix} \text{ if } \mathrm{G} = \mathrm{PGL}_2,$$

and

$$\zeta \mapsto \begin{pmatrix} 0 & -\zeta^{-1} \\ \zeta & 0 \end{pmatrix} \text{ if } \mathrm{G} = \mathrm{SL}_2.$$

The elements of $\mathcal{S}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$, viewed as measures on F, are bounded, supported away from zero, while in a neighborhood of $0 \in \mathbb{A}^1$, they have a singularity which is called the Kloosterman germ, because they are smooth multiples of the measures

$$\xi \mapsto \int_{|u|^2 = |\xi|} \psi^{-1}\left(\frac{u}{\xi} + u^{-1}\right) \mathrm{d}u \cdot \mathrm{d}^\times \xi,$$

respectively,

$$\zeta \mapsto \int_{|u| \in \pm 1 + \mathfrak{p}_{\mathrm{F}}} \psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right) \mathrm{d}u \cdot |\zeta|\,\mathrm{d}^\times \zeta.$$

For more details, we refer the reader to [Sak13a, Sak21, Sak22a].

Now let us consider the twisted-push forward of the non-standard measure $\Phi_{L(\rho,s)}$:

$$f_{L(\rho,s)}(t) := \frac{\delta_{\mathrm{B}}(t)}{1-q^{-2}} \int_{\mathrm{N}}^{*} \Phi_{L(s,\rho)}(\mathbf{w}tn)\psi^{-1}(n)\,\mathrm{d}n \cdot \mathrm{d}t.$$

We call $f_{L(\rho,s)}$ the basic vector in the Whittaker space of the L-value $L(\rho, s)$.

Specialize to ρ being a sum of copies of the standard representations of $\mathrm{G}^{\vee} = \mathrm{SL}_2$ when $\mathrm{G} = \mathrm{PGL}_2$, and the sum of copies of adjoint representations of $\mathrm{G}^{\vee} = \mathrm{PGL}_2$ when $\mathrm{G} = \mathrm{SL}_2$, and write

$$\rho = \bigoplus_{i=1}^{m} \rho_i,$$

correspondingly write $s = (s_i)_{i=1}^{m}$. We let $\mathcal{S}^{-}_{L(\rho,s)}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$ be the space of measures on $\mathfrak{C}$ which on any compact set coincide with elements of $\mathcal{S}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$, while in a neighborhood of infinity, in the above coordinates, when all of the s_i's are distinct, are of the form

$$\sum_j C_j(\zeta^{-1})|\zeta|^{1-s_i}\,\mathrm{d}^{\times}\zeta,$$

in the case $\mathrm{G} = \mathrm{SL}_2$, and

$$\sum_j C_j(\xi^{-1})|\xi|^{\frac{1}{2}-s_i}\,\mathrm{d}^{\times}\xi,$$

in the case $\mathrm{G} = \mathrm{PGL}_2$, where C_j's are locally constant functions in a neighborhood of zero. When two or more of the s_i's coincide with some complex number s_0, the corresponding summands at infinity will be replaced by

$$\left(C_1(\zeta^{-1}) + C_2(\zeta^{-1})\log|\zeta| + C_3(\zeta^{-1})\log^2|\zeta| + \cdots\right)\cdot|\zeta|^{1-s_0}\,\mathrm{d}^{\times}\zeta,$$

and similarly when $\mathrm{G} = \mathrm{PGL}_2$. According to [Sak22a], $f_{L(\rho,s)} \in \mathcal{S}^{-}_{L(\rho,s)}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$. Moreover, one can compute explicitly that

Proposition 2.2.2. *Let* $\mathrm{G} = \mathrm{PGL}_2$ *and* $\rho = \mathrm{Std}$ *be the standard representation of* $\mathrm{SL}_2(\mathbb{C})$. *For the L-value* $L(\mathrm{Std}, \mathrm{s}_1)\,L(\mathrm{Std}, \mathrm{s}_2)$, *we have the following explicit expressions of the basic vector:*

$$\Phi_{L(\mathrm{Std},\mathrm{s}_1)(\mathrm{Std},\mathrm{s}_2)} = \begin{cases} \sum_{m=0}^{\infty} \frac{q^{-m(s+\frac{1}{2})}}{1-q^{-2s}}(m+1)1_{x_m\mathrm{K}} & \textit{if} \quad s_1 = s_2 = s; \\ \sum_{m=0}^{\infty}\left(\frac{q^{-m(s_1+\frac{1}{2})}-q^{-m(s_2+\frac{1}{2})}q^{s_1-s_2}}{(1-q^{s_1-s_2})(1-q^{-(s_1+s_2)})}\right)1_{x_m\mathrm{K}} & \textit{if} \quad s_1 \neq s_2. \end{cases},$$

where

$$1_{x_m\mathrm{K}}\left(n\begin{pmatrix}\varpi_{\mathrm{F}}^{\ell} & 0\\ 0 & 1\end{pmatrix}k\right)=\begin{cases} 0 & \text{if} \quad \ell\neq m;\\ \psi(n) & \text{if} \quad \ell=m.\end{cases}$$

Here $n\in\mathrm{N}$ *and* $k\in\mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})$. *And*

- *when* $s_1=s_2=s$,

$$\frac{f_{L(\mathrm{Std,s_1})L(\mathrm{Std,s_2})}\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)}{|\xi|\,\mathrm{d}^\times\xi}=(1-q^{-2})^{-1}(1-q^{-2s})^{-1}$$
$$\cdot\begin{cases} |\xi|^{-s-\frac{1}{2}}\left((1-q^{-2s-1})\log_q|\xi|+(1-3q^{-2s-1})\right) & \text{for} \quad |\xi|\geq 1;\\ -2q^{-s-\frac{1}{2}} & \text{for} \quad |\xi|=q^{-1};\\ |\xi|^{-1}\int_{|x|^2=|\xi|}\psi^{-1}\left(\frac{x}{\xi}+x^{-1}\right)\mathrm{d}x & \text{for} \quad |\xi|<q^{-1}.\end{cases}$$

  where $\mathrm{d}^\times\xi$ *is the Haar measure such that* $\mathrm{vol}(\mathrm{T}(\mathfrak{o}_{\mathrm{F}}),\mathrm{d}^\times\xi)=1-q^{-1}$.

- *When* $s_1\neq s_2$,

$$\frac{f_{L(\mathrm{Std,s_1})L(\mathrm{Std,s_2})}\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)}{|\xi|\,\mathrm{d}^\times\xi}=(1-q^{-2})^{-1}(1-q^{s_1-s_2})^{-1}\left(1-q^{-(s_1+s_2)}\right)^{-1}$$
$$\cdot\begin{cases} |\xi|^{-s_1-\frac{1}{2}}(1-q^{-2s_1-1})-|\xi|^{-s_2-\frac{1}{2}}q^{s_1-s_2}(1-q^{-2s_2-1}) & \text{for} \quad |\xi|\geq 1;\\ -q^{-(s_1+\frac{1}{2})}+q^{-(s_2+\frac{1}{2})}q^{s_1-s_2} & \text{for} \quad |\xi|=q^{-1};\\ (1-q^{s_1-s_2})|\xi|^{-1}\int_{|x|^2=|\xi|}\psi^{-1}\left(\frac{x}{\xi}+x^{-1}\right)\mathrm{d}x & \text{for} \quad |\xi|<q^{-1}.\end{cases}$$

Proof. Let us start with the series

$$\begin{aligned} L(\mathrm{Std,s_1})\,L(\mathrm{Std,s_2}) &= \frac{1}{\det(1-q^{-s_1}\mathrm{Std})}\cdot\frac{1}{\det(1-q^{-s_2}\mathrm{Std})}\\ &= \left(\sum_{m=0}^{\infty}q^{-ms_1}\mathrm{tr}\,(\mathrm{Sym}^m\mathrm{Std})\right)\left(\sum_{m=0}^{\infty}q^{-ms_1}\mathrm{tr}\,(\mathrm{Sym}^m\mathrm{Std})\right)\\ &= \sum_{m_1,m_2=0}^{\infty}q^{-m_1s_1-m_2s_2}\mathrm{tr}\,(\mathrm{Sym}^{m_1}\mathrm{Std}\otimes\mathrm{Sym}^{\mathrm{m_2}}\mathrm{Std}).\end{aligned}$$

According to [FH91, Section 11.2],

$$\mathrm{Sym}^{m_1}\mathrm{Std}\otimes\mathrm{Sym}^{\mathrm{m_2}}\mathrm{Std}=\sum_{\ell=0}^{\min\{\mathrm{m_1,m_2}\}}\mathrm{Sym}^{\mathrm{m_1+m_2}-\ell}\mathrm{Std}.$$

Therefore, we have

$$\begin{aligned}
L(\mathrm{Std}, \mathrm{s}_1)\, L(\mathrm{Std}, \mathrm{s}_2) &= \frac{1}{\det(1-q^{-s_1}\mathrm{Std})} \cdot \frac{1}{\det(1-q^{-s_2}\mathrm{Std})} \\
&= \sum_{m_1,m_2=0}^{\infty} q^{-m_1 s_1 - m_2 s_2} \mathrm{tr} \left(\sum_{\ell=0}^{\min\{m_1,m_2\}} \mathrm{Sym}^{m_1+m_2-\ell}\mathrm{Std} \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{\ell=0}^{\infty} q^{-(m+2\ell)s_1} \sum_{m_2=\ell}^{m+\ell} q^{m_2(s_1-s_2)} \right) \mathrm{tr}\,(\mathrm{Sym}^m\mathrm{Std}) \\
&= \begin{cases} \sum_{m=0}^{\infty} \frac{q^{-ms}}{1-q^{-2s}}(m+1)\mathrm{tr}\,(\mathrm{Sym}^m\mathrm{Std}) & \text{for} \quad s_1 = s_2 = s; \\ \sum_{m=0}^{\infty} \left(\frac{q^{-ms_1} - q^{-ms_2} q^{s_1-s_2}}{(1-q^{s_1-s_2})(1-q^{-(s_1+s_2)})} \right) \mathrm{tr}\,(\mathrm{Sym}^m\mathrm{Std}) & \text{for} \quad s_1 \neq s_2. \end{cases}
\end{aligned}$$

The Casselman-Shalika formula tells us

$$\mathrm{Sat}^{-1}(\mathrm{Sym}^m\mathrm{Std}) \star 1_{\mathrm{N}\backslash\mathrm{G}(\mathfrak{o}_\mathrm{F})} = \mathrm{q}^{-\frac{\mathrm{m}}{2}} 1_{\mathrm{x_m K}}.$$

Therefore,

$$\Phi_{L(\mathrm{Std},\mathrm{s}_1)(\mathrm{Std},\mathrm{s}_2)} = \begin{cases} \sum_{m=0}^{\infty} \frac{q^{-m(s+\frac{1}{2})}}{1-q^{-2s}}(m+1)1_{x_m\mathrm{K}} & \text{if} \quad s_1 = s_2 = s; \\ \sum_{m=0}^{\infty} \left(\frac{q^{-m(s_1+\frac{1}{2})} - q^{-m(s_2+\frac{1}{2})} q^{s_1-s_2}}{(1-q^{s_1-s_2})(1-q^{-(s_1+s_2)})} \right) 1_{x_m\mathrm{K}} & \text{if} \quad s_1 \neq s_2. \end{cases}$$

According to [Sak13a, Section 6.3], we have that

$$\begin{aligned}
&\int_{\mathrm{N}} 1_{x_m\mathrm{K}} \left(\mathbf{w} \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} n \right) \psi^{-1}(n)\, \mathrm{d}n \\
&\qquad = \begin{cases} 1 & \text{if} \quad |\xi| = q^m; \\ -1 & \text{if} \quad |\xi| = q^{m-2}, m \geq 1; \\ |\xi|^{-1} \int_{|x|^2=|\xi|} \psi^{-1}\left(\frac{x}{\xi} + x^{-1}\right) \mathrm{d}x & \text{if} \quad |\xi| < 1, m = 0. \end{cases}
\end{aligned}$$

Combine all these, in the case that $s_1 = s_2 = s$, we have

$$\begin{aligned}
&\frac{f_{L(\mathrm{Std},\mathrm{s}_1)L(\mathrm{Std},\mathrm{s}_2)}\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right)}{|\xi|\, \mathrm{d}^\times \xi} \\
&= (1-q^{-2})^{-1}(1-q^{-2s})^{-1} \begin{cases} |\xi|^{-s-\frac{1}{2}} \left((1-q^{-2s-1}) \log_q |\xi| + (1-3q^{-2s-1}) \right) & \text{for} \quad |\xi| \geq 1; \\ -2q^{-s-\frac{1}{2}} & \text{for} \quad |\xi| = q^{-1}; \\ |\xi|^{-1} \int_{|x|^2=|\xi|} \psi^{-1}\left(\frac{x}{\xi} + x^{-1}\right) \mathrm{d}x & \text{for} \quad |\xi| < q^{-1}. \end{cases}
\end{aligned}$$

And in the case that $s_1 \neq s_2$, we have

$$\frac{f_{L(\mathrm{Std},s_1)L(\mathrm{Std},s_2)}\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)}{|\xi|\,\mathrm{d}^\times\xi} = (1-q^{-2})^{-1}(1-q^{s_1-s_2})^{-1}\left(1-q^{-(s_1+s_2)}\right)^{-1}$$

$$\cdot\begin{cases} |\xi|^{-s_1-\frac{1}{2}}(1-q^{-2s_1-1})) - |\xi|^{-s_2-\frac{1}{2}}q^{s_1-s_2}(1-q^{-2s_2-1}) & \text{for} \quad |\xi|\geq 1;\\ -q^{-(s_1+\frac{1}{2})}+q^{-(s_2+\frac{1}{2})}q^{s_1-s_2} & \text{for} \quad |\xi| = q^{-1};\\ (1-q^{s_1-s_2})|\xi|^{-1}\int_{|x|^2=|\xi|}\psi^{-1}\left(\frac{x}{\xi}+x^{-1}\right)\mathrm{d}x & \text{for} \quad |\xi| < q^{-1}.\end{cases}$$

□

Proposition 2.2.3. *Let* $\mathrm{G}=\mathrm{SL}_2$ *and* $\rho = \mathrm{Ad}$ *be the adjoint representation of* $\mathrm{PGL}_2(\mathbb{C})$*. For the L-value* $L(\mathrm{Ad},s)$*, we have the following explicit expressions of the basic vector:*

$$\Phi_{L(\mathrm{Ad},s)} = \sum_{m=0}^{\infty}\frac{q^{-m(s+1)}}{1-q^{-2s}}1_{x_m\mathrm{K}},$$

where

$$1_{x_m\mathrm{K}}\left(n\begin{pmatrix}\varpi_{\mathrm{F}}^{\ell} & 0\\ 0 & \varpi_{\mathrm{F}}^{-\ell}\end{pmatrix}k\right) = \begin{cases} 0 & \textit{if} \quad \ell\neq m;\\ \psi(n) & \textit{if} \quad \ell = m.\end{cases}$$

Here $n\in\mathrm{N}$ *and* $k\in\mathrm{K}$*. And*

$$\frac{f_{L(\mathrm{Ad},s)}\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right)}{|\zeta|^2\,\mathrm{d}^\times\zeta}$$

$$= (1-q^{-2})^{-1}(1-q^{-2s})^{-1}\begin{cases} |\zeta|^{-s-1}(1-q^{-s-1}) & \textit{for} \quad |\zeta|\geq 1;\\ |\zeta|^{-1}\int_{x\in\pm 1+\mathfrak{p}_{\mathrm{F}}}\psi^{-1}\left(\frac{x+x^{-1}}{\xi}\right)\mathrm{d}x & \textit{for} \quad |\zeta|\leq q^{-1}.\end{cases}$$

Proof. Let ρ_k be the descent of $\mathrm{Sym}^{2k}\mathrm{Std}$ of $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{PGL}_2(\mathbb{C})$. Then according to [Rud90], we have

$$\mathrm{Sym}^m\mathrm{Ad} = \sum_{k\leq m, k\equiv m \mod 2}\rho_k.$$

Therefore,

$$L(\mathrm{Ad},s) = \frac{1}{\det(1-q^{-s}\mathrm{Ad})} = \sum_{m=0}^{\infty}q^{-ms}\,\mathrm{Sym}^m\mathrm{Ad} = \sum_{m=0}^{\infty}\left(\sum_{k=m,k\equiv m \mod 2}^{\infty}q^{-ks}\right)\rho_m$$

$$= \sum_{m=0}^{\infty}\frac{q^{-ms}}{1-q^{-2s}}\rho_m.$$

Using the Casselman-Shalika formula again, we have

$$\mathrm{Sat}^{-1}(\rho_m) \star 1_{\mathrm{N}\backslash\mathrm{G}(\mathfrak{o}_\mathrm{F})} = q^{-m} 1_{x_m\mathrm{K}}.$$

Therefore,

$$\Phi_{L(\mathrm{Ad},s)} = \sum_{m=0}^{\infty} \frac{q^{-m(s+1)}}{1-q^{-2s}} 1_{x_m\mathrm{K}}.$$

Similar to the computation in [Sak13a], it is easy to see that

$$\begin{aligned}
&\int_{\mathrm{N}} 1_{x_m\mathrm{K}} \left(\mathbf{w} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} n \right) \psi^{-1}(n)\, \mathrm{d}n \\
&= \begin{cases} 1 & \text{if} \quad |\zeta| = q^m; \\ -1 & \text{if} \quad |\zeta| = q^{m-1}, m \geq 1; \\ |\zeta|^{-1} \int_{x\in\pm 1+\mathfrak{p}_\mathrm{F}} \psi^{-1}\left(\frac{x+x^{-1}}{\xi}\right) \mathrm{d}x & \text{if} \quad |\xi| < 1, m = 0. \end{cases}
\end{aligned}$$

Combine them, and we have

$$\begin{aligned}
&\frac{f_{L(\mathrm{Ad},s)}\left(\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right)}{|\zeta|^2\, \mathrm{d}^\times \zeta} \\
&= (1-q^{-2})^{-1}(1-q^{-2s})^{-1} \begin{cases} |\zeta|^{-s-1}(1-q^{-s-1}) & \text{for} \quad |\zeta| \geq 1; \\ |\zeta|^{-1} \int_{x\in\pm 1+\mathfrak{p}_\mathrm{F}} \psi^{-1}\left(\frac{x+x^{-1}}{\xi}\right) \mathrm{d}x & \text{for} \quad |\zeta| \leq q^{-1}. \end{cases}
\end{aligned}$$

□

2.3 Smooth Transfer for Rank One Spherical Varieties

2.3.1 Transfer Operators from Geometry

Let X be a spherical variety of G. The list of spherical varieties of rank 1 consists of a finite number of families, classified by Ahiezer [Ahi83]; see also [KVS06] and [Was96]. Up to an action of the center, the affine homogeneous spherical varieties over an algebraically closed field in characteristic 0 are listed in the following table :

	X	$\mathrm{G}_{\mathrm{X}}^{\vee}$	L_{X}
A_n	$\mathrm{GL}_n\backslash\mathrm{PGL}_{n+1}$	SL_2	$L\left(\mathrm{Std},\frac{\mathrm{n}}{2}\right)^2$
B_n	$\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}$	SL_2	$L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)$
C_n	$\mathrm{Sp}_{2n-2}\times\mathrm{Sp}_2\backslash\mathrm{Sp}_{2n}$	SL_2	$L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\mathrm{n}-\frac{3}{2}\right)$
D_n	$\mathrm{SO}_{2n-1}\backslash\mathrm{SO}_{2n}$	PGL_2	$L(\mathrm{Ad},n-1)$
B_3''	$\mathrm{G}_2\backslash\mathrm{Spin}_7$	PGL_2	$L(\mathrm{Ad},3)$
D_4''	$\mathrm{Spin}_7\backslash\mathrm{Spin}_8$	PGL_2	$L(\mathrm{Ad},n-1)$
F_4	$\mathrm{Spin}_9\backslash\mathrm{F}_4$	SL_2	$L\left(\mathrm{Std},\frac{11}{2}\right)L\left(\mathrm{Std},\frac{5}{2}\right)$
G_2	$\mathrm{SL}_3\backslash\mathrm{G}_2$	SL_2	$L\left(\mathrm{Std},\frac{5}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)$

. (2.5)

We will work over a non-Archimedean local field of characteristic 0, and will only consider the split cases.

For each spherical variety, L_{X} stands for a $\frac{1}{2}\mathbb{Z}$-graded representation $\rho=\bigoplus_{n\in\frac{1}{2}\mathbb{Z}}\rho_n$ of its dual group $\mathrm{G}_{\mathrm{X}}^{\vee}$, which is called the L-value asssociated to X, thinking of the L-value

$$\prod_n L(\rho_n\circ\varphi,n)$$

attached to any Langlands parameter φ into $\mathrm{G}_{\mathrm{X}}^{\vee}$. And the L-value here is determined in terms of the geometry of X.

Let $\pi:\mathrm{X}\times\mathrm{X}\to\mathrm{X}\times\mathrm{X}/\!\!/\mathrm{G}$ be the canonical quotient map to the affine, invariant-theoretic quotient $\mathrm{SpecF}[\mathrm{X}\times\mathrm{X}]^{\mathrm{G}}$, and the image $\pi_*\mathcal{S}(\mathrm{X}\times\mathrm{X})$ of the push-forward measure will be denoted by $\mathcal{S}(\mathrm{X}\times\mathrm{X}/\mathrm{G})$. Y. Sakellaridis proved the following theorem in [Sak21]:

Theorem 2.3.1 (Theorem 1.3.1 in [Sak21])**.** *Let $\mathfrak{C}_{\mathrm{X}}=(\mathrm{X}\times\mathrm{X})/\!\!/\mathrm{G}$. There is an isomorphism $\mathfrak{C}_{\mathrm{X}}\cong\mathbb{A}^1$, and the map $\mathrm{X}\times\mathrm{X}\to\mathbb{A}^1$ is smooth away from the preimage of two points of $\mathbb{A}^1$, that we will call singular. We fix the isomorphism as follows:*

- *When $\mathrm{G}_{\mathrm{X}}^{\vee}=\mathrm{SL}_2$, we take the set of singular points to be $\{0,1\}$, with $\mathrm{X}^{\mathrm{diag}}\subset\mathrm{X}\times\mathrm{X}$ mapping to $1\in\mathfrak{C}_{\mathrm{X}}\cong\mathbb{A}^1$.*

- *When $\mathrm{G}_{\mathrm{X}}^{\vee}=\mathrm{PGL}_2$, we take the set of singular points to be $\{-2,2\}$, with $\mathrm{X}^{\mathrm{diag}}\subset\mathrm{X}\times\mathrm{X}$ mapping to $2\in\mathfrak{C}_{\mathrm{X}}\cong\mathbb{A}^1$.*

Then there is a continuous linear isomorphism:

$$\mathcal{T} : \mathcal{S}^{-}_{L_{\mathrm{X}}}(\mathrm{N}, \psi\backslash \mathrm{G}^*/\mathrm{N}, \psi) \to \mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G}),$$

given by the following formula:

- *When* $\mathrm{G}^\vee_{\mathrm{X}} = \mathrm{SL}_2$ *with* $L_{\mathrm{X}} = L(\mathrm{Std}, \mathrm{s}_1)\mathrm{L}(\mathrm{Std}, \mathrm{s}_2)$, $s_1 \geq s_2$,
$$\mathcal{T}f(\xi) = |\xi|^{s_1 - \frac{1}{2}} \left(|\cdot|^{\frac{1}{2} - s_1} \psi(\cdot)\, \mathrm{d}\cdot \right) \star \left(|\cdot|^{\frac{1}{2} - s_2} \psi(\cdot)\, \mathrm{d}\cdot \right) \star f(\xi).$$

- *When* $\mathrm{G}^\vee_{\mathrm{X}} = \mathrm{PGL}_2$ *with* $L_{\mathrm{X}} = L(\mathrm{Ad}, s_0)$,
$$\mathcal{T}f(\zeta) = |\zeta|^{s_0 - 1} \left(|\cdot|^{1 - s_0} \psi(\cdot)\, \mathrm{d}\cdot \right) \star f(\zeta).$$

Let us conclude this section by reformulating the above convolution operators in terms of additive Fourier transforms of functions:

Lemma 2.3.2. *Write* $f(\xi) = \phi(\xi)\, \mathrm{d}\xi$ *or* $\phi(\zeta)\, \mathrm{d}\zeta$, *then*

- *When* $\mathrm{G}^\vee_{\mathrm{X}} = \mathrm{SL}_2$ *with* $L_{\mathrm{X}} = L(\mathrm{Std}, \mathrm{s}_1)\mathrm{L}(\mathrm{Std}, \mathrm{s}_2)$, $s_1 \geq s_2$,
$$\mathcal{T}f(\xi) = \left(\mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot|^{1 + s_1 - s_2} \circ \mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot|^{s_2 + \frac{1}{2}} \right)(\phi)(\xi)\, \mathrm{d}\xi.$$

- *When* $\mathrm{G}^\vee_{\mathrm{X}} = \mathrm{PGL}_2$ *with* $L_{\mathrm{X}} = L(\mathrm{Ad}, s_0)$,
$$\mathcal{T}f(\zeta) = (\mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot|^{s_0})(\phi)(\zeta)\, \mathrm{d}\zeta.$$

Remark 2.3.3. *In [Sak21], Y. Sakellaridis constructed the transfer operators through a meticulous analysis of the geometric structure of* $\mathrm{X} \times \mathrm{X}/\mathrm{G}$. *By characterizing the asymptotic behavior of the push-forward measures on* $\mathcal{S}(\mathrm{X} \times \mathrm{X}/\mathrm{G})$, *he established a rigorous comparison with the non-standard test measures for the Kuznetsov formula.*

The transfer operator is the correct operator of functoriality between the relative trace formula for X *and the Kuznetsov formula, supported by several key pieces of evidence. It was shown in [Sak13a, Sak19, Sak22a] that in the cases* A_1 *and* D_2, *it satisfies the appropriate fundamental lemma for the Hecke algebra, and it pulls back relative characters to relative characters. The same is true for the general case of* D_n *due to the work of R. Rallis [Ral82] and W. Gan–X. Wan [GW21]. The statements are essentially checked for general* A_n *in [SV17] using unfolding. There is a recent preprint of N. Le–B. Wang [LW25] deals with the* F_4 *case.*

2.3.2 Transfer Operators from Weil Representations

Let $\mathbb{V}$ and $\mathbb{W}$ be two right $\mathbb{D}$-modules, which are finite dimensional over F. We will denote the set of right $\mathbb{D}$-module morphisms by

$$\mathrm{Hom}_{\mathbb{D}}(\mathbb{V},\mathbb{W}) := \{T : \mathbb{V} \to \mathbb{W} \mid T(v_1 a + v_2 b) = T(v_1)a + T(v_2)b,\ \forall v_1, v_2 \in \mathbb{V}, a, b \in \mathbb{D}\}.$$

If $\mathbb{V} = \mathbb{W}$, we will denote this set by $\mathrm{End}_{\mathbb{D}}(\mathbb{V})$. Set $\mathrm{GL}_{\mathbb{D}}(\mathbb{V})$ to be the set of elements T in $\mathrm{End}_{\mathbb{D}}(\mathbb{V})$ such that T is invertible.

Definition 2.3.4. *Let $\epsilon = \pm 1$. We say that $(\mathbb{V}, \mathbb{B})$ is a right ϵ-Hermitian $\mathbb{D}$-module, if $\mathbb{V}$ is a right $\mathbb{D}$-module, and $\mathbb{B} : \mathbb{V} \times \mathbb{V} \to \mathbb{D}$ is a map such that*

(1) $\mathbb{B}$ is sesquilinear, i.e., for all $v_1, v_2, v_3 \in \mathbb{V}$, $a, v \in \mathbb{D}$,

$$\mathbb{B}(v_1, v_2 a + v_3 b) = \mathbb{B}(v_1, v_2)a + \mathbb{B}(v_1, v_3)b$$

and

$$\mathbb{B}(v_1 a + v_2 b, v_3) = \overline{a}\mathbb{B}(v_1, v_3) + \overline{b}\mathbb{B}(v_2, v_3).$$

(2) $\mathbb{B}$ is ϵ-Hermitian, i.e., for all $v_1, v_2 \in \mathbb{V}$,

$$\mathbb{B}(v_1, v_2) = \epsilon\overline{\mathbb{B}(v_2, v_1)}.$$

(3) $\mathbb{B}$ is non-degenerate.

We call $\mathbb{B}$ an ϵ-Hermitian form. Usually, 1-Hermitian $\mathbb{D}$-modules (resp. forms) are also called Hermitian modules (resp. forms), and -1-Hermitian modules (resp. forms) are called skew-Hermitian modules (resp. forms).

Given a right ϵ-Hermitian $\mathbb{D}$-module $(\mathbb{V}, \mathbb{B})$, we write

$$\mathrm{U}(\mathbb{V}, \mathbb{B}) := \{g \in \mathrm{GL}_{\mathbb{D}}(\mathbb{V}) \mid \mathbb{B}(gv_1, gv_2) = \mathbb{B}(v_1, v_2),\ \forall v_1, v_2 \in \mathbb{V}\}$$

for the unitary group of $(\mathbb{V}, \mathbb{B})$. Then $\mathrm{U}(\mathbb{V}, \mathbb{B})$ acts on $\mathbb{V}$ on the left. When $\mathbb{B}$ is clear from the context, we will also write $\mathrm{U}(\mathbb{V})$ for simplicity.

Given a right ϵ-Hermitian $\mathbb{D}$-module $(\mathbb{V}, \mathbb{B})$, we can construct a left $\mathbb{D}$-module $\overline{\mathbb{V}}$ together with $\overline{\mathbb{B}} : \overline{\mathbb{V}} \times \overline{\mathbb{V}} \to \mathbb{D}$ in the following way: as a set, $\overline{\mathbb{V}}$ is the set of symbols

$\{\overline{v} \mid v \in \mathbb{V}\}$. Then we give $\overline{\mathbb{V}}$ a left $\mathbb{D}$-module structure by setting, for all $v_1, v_2 \in \mathbb{V}$, $a \in \mathbb{D}$,

$$\overline{v_1} + \overline{v_2} := \overline{v_1 + v_2},\ a\overline{v} := \overline{v\overline{a}},$$

and

$$\overline{\mathbb{B}} : \overline{\mathbb{V}} \times \overline{\mathbb{V}} \to \mathbb{D} : (\overline{v_1}, \overline{v_2}) \mapsto \overline{\mathbb{B}(v_2, v_1)}.$$

Let us briefly recall the classification of (skew-)Hermitian modules and the corresponding unitary groups.

Example 2.3.5. *If $\mathbb{D} = \mathrm{F}$, the Hermitian $\mathbb{D}$-modules are classfied by the non-degenerate quadratic forms over F, which are determined by the invariants: the discriminant $\mathrm{disc}(\mathbb{V}) \in \mathrm{F}^\times/(\mathrm{F}^\times)^2$, the Hasse-Witt invariant $\epsilon(\mathbb{V}) = \pm 1$, and the dimension $\dim_{\mathrm{F}}(\mathbb{V})$. For more details, see [Ser73]. In this case, we have $\mathrm{U}(\mathbb{V}, \mathbb{B}) = \mathrm{O}(\mathbb{V}, \mathbb{B})$. When $\mathbb{V} = \mathrm{F}^n$, and $\mathbb{B}$ is given by*

$$\mathrm{F}^n \times \mathrm{F}^n : \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) \mapsto \sum_{i=1}^{n} x_i y_{n+1-i},$$

we call it the standard Hermitian model of rank n over F. When $n = 2m+1$ is odd, we will add one more $x_{m+1}y_{m+1}$ for the purpose of the paper.

The skew-Hermitian forms are classified by non-degenerate alternative forms over F, which is unique up to isomorphism, see [GW09, Lemma 1.1.5]. For the purpose of this paper, when $\mathbb{W} = \mathrm{F}^2$, and $\mathbb{B}$ is given by

$$\mathrm{F}^2 \times \mathrm{F}^2 \to \mathrm{F} : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto x_1 y_2 - x_2 y_1,$$

we call it the standard skew-Hermitian model of rank 2 over F. We also write $e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and call $\mathbb{W} = e\mathbb{D} + f\mathbb{D}$ the standard polarization.

Example 2.3.6. *If $\mathbb{D} = \mathrm{F} \times \mathrm{F}$. Since $\mathbb{V}$ is an $\mathrm{F} \times \mathrm{F}$-module, and hence has the form $\mathrm{V} \times \mathrm{V}^\vee$, for some F-vector space V and its F-dual space $\mathrm{V}^\vee$. Up to isomorphism, any Hermitian $\mathbb{D}$-module is isomorphic to the one defined by*

$$\mathbb{B} : \big((v_1, v_1^\vee), (v_2, v_2^\vee)\big) \mapsto (\langle v_1, v_2^\vee \rangle, \langle v_2, v_1^\vee \rangle) \in \mathbb{D} = \mathrm{F} \times \mathrm{F},$$

where $\langle \cdot,\cdot\rangle : \mathrm{V}\times\mathrm{V}^\vee \to \mathrm{F}$ *is the canonical pair between* V *and* $\mathrm{V}^\vee$.

In this case, it is easy to see

$$\mathrm{U}(\mathbb{V},\mathbb{B}) \cong \mathrm{GL}(\mathrm{V}),$$

where the embedding is given by $\mathrm{GL}(\mathrm{V}) \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1}\end{pmatrix}$ *in terms of matrices. We also fix a basis of* V *and its dual basis, then* $\mathrm{V}=\mathrm{F}^n$ *and* $\mathbb{B}$ *is given by*

$$\left(\left(\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix},\begin{pmatrix} y_1\\ \vdots \\ y_n\end{pmatrix}\right),\left(\begin{pmatrix} \xi_1\\ \vdots \\ \xi_n\end{pmatrix},\begin{pmatrix} \zeta_1\\ \vdots \\ \zeta_n\end{pmatrix}\right)\right)\mapsto\left(\sum_{i=1}^n x_i\zeta_i, \sum_{i=1}^n y_i\xi_i\right).$$

We call it the standard Hermitian model of rank n *over* $\mathrm{F}\times\mathrm{F}$.

Similarly, any skew Hermitian space is of the form $\mathbb{W}=\mathrm{W}\times\mathrm{W}^\vee$ *with*

$$\mathbb{B} : \big((w_1,w_1^\vee),(w_2,w_2^\vee)\big)\mapsto (\langle w_1,w_2^\vee\rangle, -\langle w_2,w_1^\vee\rangle)\in\mathbb{D}=\mathrm{F}\times\mathrm{F},$$

up to isomorphism. Then $\mathrm{U}(\mathbb{W},\mathbb{B})\cong\mathrm{GL}(\mathrm{W})$ *as well. In the special case that* $\mathbb{W}=\mathrm{W}\times\mathrm{W}^\vee$ *for a two dimensional vector space* W *over* F, *we have* $\mathrm{U}(\mathbb{W},\mathbb{B})\cong\mathrm{GL}_2$. *Choose a basis* $\{e_1,e_2\}$ *of* W *and its dual basis* $\{e_1^\vee,e_2^\vee\}\in\mathrm{W}^\vee$, *and let*

$$e=(e_1,e_2^\vee),\ f=(-e_2,e_1^\vee),$$

then we have $\mathbb{B}(e,f)=1$ *and* $\mathbb{B}(e,e)=\mathbb{B}(f,f)=0$. *We call* $\mathbb{W}$ *the standard skew-Hermitian model of rank* 2 *over* $\mathrm{F}\times\mathrm{F}$ *and* $\mathbb{W}=e\mathbb{D}+f\mathbb{D}$ *the standard polarization of* $\mathbb{W}$.

Example 2.3.7. *If* $\mathbb{D}=\mathrm{Mat}_2(\mathrm{F})$. *According to the structure theory of simple central algebras,* $\mathbb{V}$ *has the form*

$$\mathrm{F}^2\oplus\mathrm{F}^2\oplus\cdots\oplus\mathrm{F}^2\cong\mathrm{F}^n\otimes_{\mathrm{F}}\mathrm{F}^2,$$

where we view F^2 *as the space of row vectors, and* $\mathrm{Mat}_2(\mathrm{F})$ *acts on the right.*

Let $\mathbb{B}$ *be an* ϵ*-Hermitian form on* $\mathbb{V}$, *then* $\mathbb{B}$ *defines a map* $\sigma_{\mathbb{B}} : \mathrm{End}_{\mathbb{D}}(\mathbb{V})\to\mathrm{End}_{\mathbb{D}}(\mathbb{V})$, *which is called the adjoint involution, characterized by*

$$\mathbb{B}(v_1,f(v_2))=\mathbb{B}(\sigma_{\mathbb{B}}(f)(v_1),v_2),\ \forall v_1,v_2\in\mathbb{V},\ f\in\mathrm{End}_{\mathbb{D}}(\mathbb{V}).$$

This map $\mathbb{B} \mapsto \sigma_{\mathbb{B}}$ is a one-to-one correspondence between ϵ-Hermitian forms on $\mathbb{V}$ up to a scalar in $\mathrm{F}^{\times}$, and involutions σ on $\mathrm{End}_{\mathbb{D}}(\mathbb{V})$ such that $\sigma(x) = x$ for all $x \in \mathrm{F}$. See [KMRT98, Chapter 1, Section 4]. Accoding to the Schur-Weyl duality, we know $\mathrm{End}_{\mathbb{D}}(\mathbb{V}) = \mathrm{End}_{\mathrm{F}}(\mathrm{F}^n) = \mathrm{Mat}_n(\mathrm{F})$, acting on $\mathbb{V} = \mathrm{F}^n \otimes_{\mathrm{F}} \mathrm{F}^2$ on the left. Then the ϵ-Hermitian forms are classified by convolutions on Mat_n preserving F.

Let us consider the special case that $\mathbb{V} = \mathbb{D}^n$, where $\mathbb{D}$ acts on the right. Consider the Hermitian form

$$\mathbb{B} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{D} : ((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \mapsto \sum_{i=1}^{n} \overline{x_i} y_i.$$

If we identify $\mathbb{D} \cong \mathrm{F}^2 \otimes_{\mathrm{F}} \mathrm{F}^2$, and $\mathbb{D}^n \cong \mathrm{F}^{2n} \otimes_{\mathrm{F}} \mathrm{F}^2$ as above, then the above form is given by

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_{2n} \end{pmatrix} \otimes (x_1, x_2), \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} \otimes (y_1, y_2) \right)$$
$$\mapsto (v_1, \cdots, v_{2n}) \cdot \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} \cdot J^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (y_1, y_2),$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. At this time, we have $\mathrm{U}(\mathbb{V}, \mathbb{B}) = \mathrm{Sp}_{2n}$, where Sp_{2n} is defined by the symplectic form given by $\mathrm{diag}(J, \cdots, J)$, and Sp_{2n} acts on $\mathbb{D}^n = \mathrm{F}^{2n} \otimes_{\mathrm{F}} \mathrm{F}^2$ on the left. And we call it the standard Hermitian model of rank n over $\mathbb{D}$.

Another special case is that $\mathbb{W} = \mathbb{D}^2$, and consider the skew-Hermitian form

$$\mathbb{B} : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{D} : ((x_1, x_2), (y_1, y_2)) \mapsto \overline{x_1} y_2 - \overline{x_2} y_1.$$

Then identify $\mathbb{D} \cong \mathrm{F}^2 \otimes_{\mathrm{F}} \mathrm{F}^2$, and $\mathbb{D}^n \cong \mathrm{F}^{2n} \otimes_{\mathrm{F}} \mathrm{F}^2$ as above, this form is given by

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_4 \end{pmatrix} \otimes (x_1, x_2), \begin{pmatrix} w_1 \\ \vdots \\ w_4 \end{pmatrix} \otimes (y_1, y_2) \right)$$

$$\mapsto (v_1, \cdots, v_4) \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_4 \end{pmatrix} \cdot J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (y_1, y_2),$$

and hence $\mathrm{U}(\mathbb{W}, \mathbb{B}) = \mathrm{O}_4$. Let $e = (\mathrm{I}_2, 0) \in \mathbb{D}^2$ and $f = (0, \mathrm{I}_2) \in \mathbb{D}^2$, then we have $\mathbb{B}(e, f) = 1$, and $\mathbb{B}(e, e) = \mathbb{B}(f, f) = 0$. In this case, we call it the standard skew-Hermitian model of rank 2 over $\mathbb{D}$.

Let $(\mathbb{V}, \mathbb{B}_{\mathbb{V}})$ be a Hermitian $\mathbb{D}$-module, and $(\mathbb{W}, \mathbb{B}_{\mathbb{W}})$ be a skew Hermitian $\mathbb{D}$-module. On the F-vector space $\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}$, one can define an alternating form $\mathbb{B}$ by setting

$$\mathbb{B}(v_1 \otimes w_1, v_2 \otimes w_2) := \tau(\mathbb{B}_{\mathbb{V}}(v_1, v_2) \cdot \overline{\mathbb{B}_{\mathbb{W}}}(\overline{w_2}, \overline{w_1})) \in \mathrm{F}$$

for $v_1, v_2 \in \mathbb{V}$ and $w_1, w_2 \in \overline{\mathbb{W}}$. Let

$$\mathrm{Sp}(\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}, \mathbb{B})$$

be the symplectic group for the above alternating form. Moreover, there is a natural map

$$\mathrm{U}(\mathbb{V}, \mathbb{B}_{\mathbb{V}}) \times \mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) \to \mathrm{Sp}(\mathrm{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}, \mathbb{B}) : (g_1, g_2) \mapsto (v \otimes \overline{w} \mapsto g_1 v \otimes \overline{g_2 w}).$$

Let $\mathbb{H} = \mathbb{H}(\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}) = \mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}} \ltimes \mathrm{F}$ be the Hsisenberg group associated with $\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}$.

Assume now $\mathbb{V}$ is the standard Hermitian model of rank n over $\mathbb{D}$, $\mathbb{W}$ is the standard skew-Hermitian model of rank 2 over $\mathbb{D}$, and $\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}} = \mathbb{X} \oplus \mathbb{Y}$ be the complete polarization that corresponds to the standard polarization $\mathbb{W} = e\mathbb{D} \oplus f\mathbb{D}$. Let $(\omega_{\psi,\mathbb{X},\mathbb{Y}}, \mathcal{F}(\mathbb{Y}))$ be the Schrodinger model of the Weil representation of $\mathbb{H}$. According to Stone-von-Neumann theorem, for each $g \in \mathrm{Sp}(\mathbb{V} \otimes_{\mathbb{D}} \overline{\mathbb{W}}, \mathbb{B})$, there is an isomorphism $M = M(g) \in \mathrm{GL}(\mathcal{F}(\mathbb{Y}))$ of $\mathcal{F}(\mathbb{Y})$ such that

$$M \circ \omega_{\psi,\mathbb{X},\mathbb{Y}}(h) = \omega_{\psi,\mathbb{X},\mathbb{Y}}(gh) \circ M.$$

Define

$$\begin{aligned}&\mathrm{Mp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}})^{\mathbb{C}^\times}_{\psi,\mathcal{F}(\mathbb{Y})}\\&:=\{(g,M)\in\mathrm{Sp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}},\mathbb{B})\times\mathrm{GL}(\mathcal{F}(\mathbb{Y}))\mid M\circ\omega_{\psi,\mathbb{X},\mathbb{Y}}(h)=\omega_{\psi,\mathbb{X},\mathbb{Y}}(gh)\circ M\}.\end{aligned}$$

Let $c^\psi_{\mathbb{X}}:\mathrm{Sp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}}\mathbb{B})\times\mathrm{Sp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}},\mathbb{B})\to\mu_8$ be the Leray cocycle, and define

$$\mathrm{Mp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}})^{\mathbb{C}^1}_{\psi,\mathbb{X}}:=\mathrm{Sp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}})\times\mathbb{C}^1$$

as a set but with group structure given by the Leray cocycle $c^\psi_{\mathbb{X}}$, namely

$$(g,z)\cdot(g',z'):=(gg',c^\psi_{\mathbb{X}}(g,g')zz').$$

Let $\mathbf{r}=\mathbf{r}_{\mathbb{X},\mathbb{Y}}$ be the standard Weil operator; then we have the Weil representation

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}:\mathrm{Mp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}})^{\mathbb{C}^1}_{\psi,\mathbb{X}}\to\mathrm{GL}(\mathcal{F}(\mathbb{Y})):(g,z)\mapsto z\mathbf{r}(g),$$

and the action of $\mathrm{Mp}(\mathbb{V}\otimes_{\mathbb{D}}\overline{\mathbb{W}})^{\mathbb{C}^1}_{\psi,\mathbb{X}}$ is explicitly described as follows: for any $\Phi\in\mathcal{F}(\mathbb{Y})$, and $y\in\mathbb{Y}$, we have for

-
$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\,(g,1)\,\Phi(y)=|\det g|^{\frac{1}{2}}_{\mathbb{V}}\Phi(g^{-1}y),\ \forall g\in\mathrm{U}(\mathbb{V},\mathbb{B}_{\mathbb{V}}),$$
where $|\det g|_{\mathbb{V}}=1$ except when $\mathbb{D}=\mathrm{F}$ and $\mathrm{U}(\mathbb{V},\mathbb{W}_{\mathbb{V}})=\mathrm{O}(\mathbb{V})$, in which case $|\det g|_{\mathbb{V}}=\pm1$ depending on whether g is in $\mathrm{SO}(\mathbb{V})$ or not.

-
$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\,(1,z)\,\Phi(y)=z\Phi(y),\ \forall z\in\mathbb{C}^1.$$

For the action of $\mathrm{U}(\mathbb{W},\mathbb{B}_{\mathbb{W}})$, it is given as follows:

- For $\mathbb{D}=\mathrm{F}$, we have $\mathrm{U}(\mathbb{W},\mathbb{B}_{\mathbb{W}})=\mathrm{SL}_2$, and

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix}\zeta&0\\0&\zeta^{-1}\end{pmatrix},1\right)\Phi(y)=|\zeta|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\Phi(\zeta y),\ \forall\zeta\in\mathrm{F}^\times,$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix}1&u\\0&1\end{pmatrix},1\right)\Phi(y)=\psi\left(\frac{u\cdot\mathbb{B}(y,y)}{2}\right)\Phi(y),\ \forall u\in\mathrm{F},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},1\right)\Phi(y)=\int_{\mathbb{V}}\Phi(x)\psi^{-1}(\mathbb{B}(x,y))\,\mathrm{d}x.$$

- For $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, we have $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{GL}_2$, and

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, 1\right)\Phi(y) = \left|\frac{t_1}{t_2}\right|^{\frac{\dim_{\mathrm{F}} \mathbb{V}}{4}} \Phi\left(y(t_1, t_2^{-1})\right), \ \forall t_1, t_2 \in \mathrm{F}^{\times},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, 1\right)\Phi(y) = \psi\left(\frac{u \cdot \tau(\mathbb{B}(y,y))}{2}\right)\Phi(y), \ \forall u \in \mathrm{F},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right)\Phi(y) = \int_{\mathbb{V}} \Phi(x)\psi^{-1}(\tau(\mathbb{B}(x,y)))\,\mathrm{d}x.$$

- For $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$, we have $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{O}_4$, and

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, 1\right)\Phi(v \otimes \overline{f}) = \Phi(vg \otimes \overline{f}), \ \forall g \in \mathrm{SL}_2(\mathrm{F}) \subset \mathbb{D},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}, 1\right)\Phi(v \otimes \overline{f}) = |t_1 t_2|^{\frac{\dim_{\mathrm{F}} \mathbb{V}}{4}} \Phi\left(v \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \otimes \overline{f}\right), \ \forall t_1, t_2 \in \mathrm{F}^{\times},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 1\right)\Phi(y) = \psi\left(\frac{n \cdot \tau(\mathbb{B}_{\mathbb{V}}(y,y))}{2}\right)\Phi(y), \ \forall u \in \mathrm{F},$$

$$\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, 1\right)\Phi(y) = \int_{\mathbb{V}} \Phi(x)\psi^{-1}(\tau(\mathbb{B}(x,y)))\,\mathrm{d}x.$$

For more details, see [GKT25].

In the following, for simplicity, write $\omega_\psi := \omega_{\psi,\mathbb{X},\mathbb{Y}}$. In order to analyze the Hecke algebra action, the following remarks on the splitting are in order. In the case that

$\mathbb{D} = \mathrm{F}$ and $\dim_{\mathrm{F}} \mathbb{V}$ is even, according to [GKT25, Theorem 11.6], the map

$$\mathrm{SO}(\mathbb{V}) \times \mathrm{SL}_2 \to \mathrm{Mp}(\mathbb{V} \otimes_{\mathbb{D}} \mathbb{W})^{\mathbb{C}^1}_{\psi,\mathbb{X}} : (g, h) \mapsto (g \otimes h, 1)$$

is a splitting of $\mathrm{SO}(\mathbb{V}) \times \mathrm{SL}_2$ to $\mathrm{Mp}(\mathbb{V} \otimes_{\mathbb{D}} \mathbb{W})^{\mathbb{C}^1}_{\psi,\mathbb{X}}$, hence the Weil representation is an honest representation of $\mathrm{SO}(\mathbb{V}) \times \mathrm{SL}_2$ by restriction. The same holds for the case when $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$.

In the case that $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, this map $(g, h) \mapsto (g \otimes h, 1)$ is also the Leray splitting according to [GKT25, Proposition 12.21].

The only non-trivial case is when $\mathbb{D} = \mathrm{F}$ and $\dim_{\mathrm{F}} \mathbb{V}$ is odd. In this case, let

$$\widetilde{\mathrm{SL}_2} = \mathrm{Mp}(\mathbb{W})^{\mu_2}_{\mathbb{X}} := \mathrm{Sp}(\mathbb{W}) \times \mu_2 = \mathrm{SL}_2 \times \mu_2$$

as a set, with group multiplication given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) := (g_1 g_2, \epsilon_1 \epsilon_2 c_{\mathbb{X}}(g_1, g_2)),$$

where $c_{\mathbb{X}}$ is the Rao cocycle. In the case of SL_2, recall that we have the Rao function

$$\mathbf{x}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} c & \text{if} \quad c \neq 0; \\ d & \text{if} \quad c = 0, \end{cases}$$

and the Rao cocycle is given by

$$c_{\mathbb{X}}(g_1, g_2) := (\mathbf{x}(g_1), \mathbf{x}(g_2))_{\mathrm{F}}(-\mathbf{x}(g_1)\mathbf{x}(g_2), \mathbf{x}(g_1 g_2))_{\mathrm{F}},$$

where $(\cdot, \cdot)_{\mathrm{F}}$ is the Hilbert symbol of F. Note that this $c_{\mathbb{X}}$ corresponds to the α used in [Jac87]. Then we have a group homomorphism [GKT25, Theorem 11.6]

$$\widetilde{\mathrm{SL}_2} \to \mathrm{Mp}(\mathbb{V} \otimes_{\mathbb{D}} \mathbb{W})^{\mathbb{C}^1}_{\psi,\mathbb{X}} : (g, \epsilon) \mapsto (1 \otimes g, \epsilon\gamma(\mathbf{x}(g), \psi_{\frac{1}{2}})^{-\dim_{\mathrm{F}}(\mathbb{V})}(\mathbf{x}(g), \det \mathbb{V})_{\mathrm{F}}),$$

where $\det(\mathbb{V}) = (-1)^{\frac{\dim_{\mathrm{F}} \mathbb{V} - 1}{2}}$ and $\gamma(\cdot, \psi_{\frac{1}{2}})$ is the normalized Weil index as in [GKT25, Section 3.3]. Following [Jac87], we employ the normalized cocycle β for the metaplectic group as defined in [Jac87]. To avoid confusion, we introduced the following notation to distinguish it from the conventions used in the previous sections. Write

$$\mathrm{Mp}_2 := \mathrm{SL}_2 \times \{\pm 1\}$$

as a set, with multiplication given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) := (g_1g_2, \epsilon_1\epsilon_2\beta(g_1, g_2)),$$

where $\beta(g_1, g_2) := \alpha(g_1, g_2)s(g_1)s(g_2)s(g_1g_2)^{-1}$, and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define

$$s(g) := \begin{cases} (c, d)_{\mathrm{F}} & \text{if } cd \neq 0 \text{ and } \mathrm{val}(c) \equiv 1 \mod 2; \\ 1 & \text{otherwise.} \end{cases}$$

In this case $k \mapsto (k, 1)$ is an isomorphism of $\mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})$ onto a subgroup K' of Mp_2. And $\mathrm{Mp}_2 \to \widetilde{\mathrm{SL}_2} : (g, \epsilon) \mapsto (g, \epsilon s(g))$ is a group isomorphism. We view SL_2 as a subset of Mp_2 via $g \mapsto (g, 1)$. Let $\mathcal{H}(\mathrm{Mp}_2, \mathrm{K}')$ be the spherical Hecke algebra of Mp_2, which consists of K' bi-invariant functions with compact support. Then there is an isomorphism

$$\mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})) \cong \mathcal{H}(\mathrm{Mp}_2, \mathrm{K}') : h \mapsto h'.$$

as fixed in [Jac87, Section 2], which we review in Section 2.7.3 for the reader's convenience. We will identify $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{SL}_2$ as a subset of Mp_2 using the set-theoretical section $g \mapsto (g, 1)$.

Remark 2.3.8. *For the sake of clarity, we provide a brief comparison between our notation and that employed in [Jac87]. The additive character ψ used in [Jac87] should be $\psi_{-2}(x) := \psi(-2x)$ in our notaion. We take $\epsilon = 1$ and $\eta = 1$ in [Jac87]. Note that the normalization of the Weil constant in [Jac87] differs by a factor of -2 from the convention adopted in [GKT25]. So the Weil constant in [Jac87] is $\gamma(\cdot, \psi_{(-2)(-\frac{1}{2})}) = \gamma(\cdot, \psi)$ in our notation, and the μ function in [Jac87] is still $\gamma(\cdot, \psi)(\cdot, -1)_{\mathrm{F}} = \gamma(\cdot, \psi)^{-1}$.*

For $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}})$, write

$$\mathrm{T} := \begin{cases} \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \mid \zeta \in \mathrm{F}^\times \right\} & \text{if } \mathbb{D} = \mathrm{F}; \\ \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mid t_1, t_2 \in \mathrm{F}^\times \right\} & \text{if } \mathbb{D} = \mathrm{F} \times \mathrm{F}; \\ \left\{ \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix} \mid \xi \in \mathrm{F}^\times \right\} & \text{if } \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

$$\mathrm{N} := \begin{cases} \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathrm{F} \right\} & \text{if} \quad \mathbb{D} = \mathrm{F},\ \mathrm{F} \times \mathrm{F}; \\ \left\{ \begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid u \in \mathrm{F} \right\} & \text{if} \quad \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

We identify N with $\mathbb{G}_a$ in the obvious way. And finally write

$$\mathbf{w} := \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{if} \quad \mathbb{D} = \mathrm{F},\ \mathrm{F} \times \mathrm{F}; \\ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \text{if} \quad \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

In the following, let $(\mathbb{V}, \mathbb{B}_{\mathbb{V}})$ be the standard Hermitian model of rank n and $(\mathbb{W}, \mathbb{B}_{\mathbb{W}})$ be the standard sekw-Hermitian model of rank 2. Fix the following $v_0 \in \mathbb{V}$ such that

$$\frac{\tau(\mathbb{B}_{\mathbb{V}}(v_0, v_0))}{2} = 1 \in \mathrm{F} :$$

$$v_0 := \begin{cases} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathrm{F}^n & \text{if} \quad \mathbb{D} = \mathrm{F}; \\ \left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right) \in \mathrm{V} \times \mathrm{V}^\vee = \mathrm{F}^n \times \mathrm{F}^n & \text{if} \quad \mathbb{D} = \mathrm{F} \times \mathrm{F}; \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I}_2 \end{pmatrix} \in \mathbb{D}^n & \text{if} \quad \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

Set

$$\mathrm{X} := \left\{ v \in \mathbb{V} \mid \frac{\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2} = 1 \right\} \subset \mathbb{V}.$$

Then X is a closed subvariety of $\mathbb{V}$.

Lemma 2.3.9. $\mathrm{U}(\mathbb{V})$ *acts transitively on* X. *Moreover, let*

$$v_0^{\perp} := \{ v \in \mathbb{V} \mid \mathbb{B}_{\mathbb{V}}(v, v_0) = 0 \}.$$

The stabilizer of v_0 *in* $\mathrm{U}(\mathbb{V})$ *is* $\mathrm{U}(v_0^{\perp})$, *where the Hermitian form on* $v_0^{\perp}$ *is the restriction of* $\mathbb{B}_{\mathbb{V}}$ *to* $v_0^{\perp} \times v_0^{\perp}$. *Therefore, we have*

$$\mathrm{X} \cong \begin{cases} \mathrm{SO}_n/\mathrm{SO}_{n-1} & \textit{if} \quad \mathbb{D} = \mathrm{F}; \\ \mathrm{GL}_n/\mathrm{GL}_{n-1} & \textit{if} \quad \mathbb{D} = \mathrm{F} \times \mathrm{F}; \\ \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} & \textit{if} \quad \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

Proof. Let us prove it case by case.

(1) If $\mathbb{D} = \mathrm{F}$, it is Witt's theorem. See [Ser73, Chapter 4]. In this case, we have $\mathrm{U}(\mathbb{V}) = \mathrm{O}(\mathbb{V}, \mathbb{B}_{\mathbb{V}})$ and $\mathrm{U}(v_0^{\perp}) = \mathrm{O}(v_0^{\perp}, \mathbb{B}_{\mathbb{V}})$, where we still write the restriction of $\mathbb{B}_{\mathbb{V}}$ to $v_0^{\perp} \times v_0^{\perp}$ by $\mathbb{B}_{\mathbb{V}}$. Since we can always find some element $g \in \mathrm{O}_n \setminus \mathrm{SO}_n$ such that $gv_0 = v_0$, SO_n acts on X transitively as well.

(2) If $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, and $\mathbb{V} = \mathrm{F}^n \times \mathrm{F}^n$, then we have

$$\begin{aligned} \mathrm{X} &= \{(v, v^{\vee}) \in \mathrm{F}^n \times \mathrm{F}^n \mid \mathbb{B}((v, v^{\vee}), (v, v^{\vee})) = (\langle v, v^{\vee} \rangle, \langle v, v^{\vee} \rangle) = (1,1)\} \\ &= \{(v, v^{\vee}) \in \mathrm{F}^n \times \mathrm{F}^n \mid \langle v, v^{\vee} \rangle = 1\}. \end{aligned}$$

For simplicity, write $e_i \in \mathrm{V} = \mathrm{F}^n$ for the standard vector with 1 in the i-th component and 0 otherwise for $1 \leq i \leq n$. Write $\{e_i^{\vee}\}_{i=1}^n \subset \mathrm{V}^{\vee} = \mathrm{F}^n$ for the dual basis, which can be identified with $\{e_i\}_{i=1}^n$ since we identify $\mathrm{V}^{\vee} = \mathrm{F}^n$ via the standard pair between F^n and F^n. For $g \in \mathrm{GL}_n$, if $ge_n = e_n$, then g is of the form

$$\begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & 0 \\ * & \cdots & * & 1 \end{pmatrix}.$$

Similarly, ${}^t g^{-1} e_n^\vee = e_n^\vee$ implies that g is of the form

$$\begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then $g(e_n, e_n^\vee) = (ge_n, {}^t g^{-1} e_n^\vee) = (e_n, e_n^\vee)$ implies that $g \in \mathrm{GL}_{n-1} = \mathrm{GL}((e_n^\vee)^\perp) \cong \mathrm{U}((e_n, e_n^\vee)^\perp)$. As for the transitivity, suppose $(v, v^\vee) \in \mathbb{V}$ such that $\langle v, v^\vee \rangle = 1$. Let $\{e_1', \cdots, e_{n-1}'\}$ be a basis of $\{w \in \mathrm{V} \mid \langle w, v^\vee \rangle = 0\}$. Let $g \in \mathrm{GL}_n$ be such that $ge_n = v$ and $ge_i = e_i'$ for $1 \leq i \leq n-1$. Then we have for any $1 \leq i \leq n-1$

$$\langle e_i', {}^t g^{-1} e_n^\vee \rangle = \langle g^{-1} e_i', e_n^\vee \rangle = \langle e_i, e_n^\vee \rangle = 0,$$

and

$$\langle v, {}^t g^{-1} e_n^\vee \rangle = \langle g^{-1} v, e_n^\vee \rangle = \langle e_n, e_n^\vee \rangle = 1.$$

Therefore, we have ${}^t g^{-1} e_n^\vee = v^\vee$.

(3) If $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$, write $v_0 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \otimes (1,0) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \otimes (0,1)$ under the isomorphism $\mathbb{D}^n \cong \mathrm{F}^{2n} \otimes_{\mathrm{F}} \mathrm{F}^2$. Once we identitfy $\mathrm{U}(\mathbb{V}) \cong \mathrm{Sp}_{2n}$, then the stabilizer of v_0 is

$$\left\{ g \in \mathrm{Sp}_{2n} \mid g \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, g \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\} = \mathrm{Sp}_{2n-2},$$

which is exactly $\mathrm{U}(v_0^\perp)$. To see the action is transitive, note that any $v \in \mathbb{V}$ can be written as the form $v_1 \otimes (1,0) + v_2 \otimes (0,1)$, and

$$\mathbb{B}(v, v) = {}^t v_1 \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} v_2,$$

it suffices to show that for any $v_1, v_2 \in \mathrm{F}^{2n}$ such that ${}^t v_1 \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} v_2{=}1$, there is some $g \in \mathrm{Sp}_{2n}$ such that

$$g \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} = v_1, \ g \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = v_2.$$

According to [Jac74, Section 6.1], we can extend $\{v_1, v_2\}$ to a basis $\{v_1, v_2, \cdots, v_{2n}\}$ of F^{2n} such that the symplectic matrix with respect to this basis is also given by

$$\begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}.$$

Write e_i for the i-th standard vector of F^{2n} with 1 in the i-th entry and 0 otherwise. Consider $g \in \mathrm{GL}_{2n}$ such that $ge_{2n-1} = v_1, ge_{2n} = v_2$, and $ge_i = v_{i-2}$ for $3 \leq i \leq 2n$, then we have $g \in \mathrm{Sp}_{2n}$.

□

There is a natural restriction map

$$\mathrm{res} : \mathcal{F}(\mathbb{V}) \to \mathcal{F}(\mathrm{X}),$$

which is surjective since $\mathrm{X} \hookrightarrow \mathbb{V}$ is a closed embedding. On the other hand, for any $g \in \mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}})$, g acts on $\mathcal{F}(\mathbb{V})$ via the Weil representation. Write

$$\Omega : \mathcal{F}(\mathbb{V}) \to \mathcal{C}^\infty(\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}})) : \Phi \mapsto (g \mapsto \omega_\psi(g)\Phi(v_0)).$$

The first easy lemma is

Lemma 2.3.10. *Given a function $\Phi \in \mathcal{F}(\mathbb{V})$, the following regulized integral*

$$\mathbb{T}(\Phi)(t) := \int_{\mathrm{N}}^{*} \Omega(\Phi)(\mathbf{w}tu)\psi^{-1}(u)\,\mathrm{d}u := \lim_{k \to +\infty} \int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \Omega(\Phi)(\mathbf{w}tu)\psi^{-1}(u)\,\mathrm{d}u \tag{2.6}$$

stabilize for sufficiently large k and $t \in \mathrm{T}$, where we identify N with $\mathbb{G}_a$ in the usual way. We denote this space of such functions by

$$\mathrm{Orb}_{\mathbb{V}}(\mathrm{T}).$$

Proof. First note that $\Omega(\Phi)(\mathbf{w}tu)$ differs $\omega_\psi(\mathbf{w})\omega_\psi(t)\omega_\psi(u)\Phi(v_0)$ only by a factor of t, so let us consider

$$\lim_{k\to+\infty}\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi(\mathbf{w})\omega_\psi(t)\omega_\psi(u)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u$$

in the following.

Note that for any $k \in \mathbb{Z}$, we have

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi(\mathbf{w})\omega_\psi(t)\omega_\psi(u)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}} \omega_\psi(t)\omega_\psi(u)\Phi(v)\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u\\
&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}} \omega_\psi(t)\psi\left(\frac{u\cdot\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\right)\Phi(v)\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u
\end{aligned}$$

Since $v\mapsto\omega_\psi(t)\psi\left(\frac{u\cdot\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\right)\Phi(v)$ is a Schwartz function on $\mathbb{V}$, we are allowed to change the order in the integration. Moreover, in all cases, it can be checked easily that there are $x(t), y(t) \in \mathrm{F}^\times$, $n(\mathbb{V}) \in \mathbb{Z}$ and a smooth map $z : \mathrm{T}\times\mathbb{V}\to\mathbb{V}$ such that

$$\omega_\psi(t)\psi\left(\frac{u\cdot\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\right)\Phi(v)=|y(t)|^{n(\mathbb{V})}\psi\left(\frac{u\cdot x(t)\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\right)\Phi(z(t,v)).$$

Then the above integral becomes

$$\int_{\mathbb{V}}|y(t)|^{n(\mathbb{V})}\Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\frac{x(t)\tau(\mathbb{B}(v,v))}{2}-1\right)\right)\mathrm{d}u\,\mathrm{d}v.$$

Note that the inner integral

$$\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\frac{x(t)\tau(\mathbb{B}(v,v))}{2}-1\right)\right)\mathrm{d}u$$

is non-zero only if

$$\frac{\tau(\mathbb{B}(v,v))}{2}\in x(t)^{-1}(1+\mathfrak{p}_{\mathrm{F}}^{k}),$$

in which case, we have

$$\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \psi\left(u\left(\frac{x(t)\tau(\mathbb{B}(v,v))}{2}-1\right)\right)\,\mathrm{d}u = q^k.$$

Consider the map $\mathbb{V} \to \mathrm{F} : w \mapsto \dfrac{\tau(\mathbb{B}(w,w))}{2}$, which is a submersion outside the zero vector. According to [Igu00, Chapter 7], the Haar measure $\mathrm{d}v$ on $\mathbb{V}$ and $\mathrm{d}x$ on F induce measures $|\omega_x|$ on each fiber

$$\mathbb{V}_x := \left\{ v \in \mathbb{V} \mid \frac{\tau(\mathbb{B}(v,v))}{2} = x \right\},$$

such that for any $\Phi \in \mathbb{V}$,

$$\int_{\mathbb{V}} \Phi(v)\,\mathrm{d}v = \int_{\mathrm{F}} \int_{\mathbb{V}_x} \Phi(v)|\omega_x(v)|x.$$

Then the desired integral is

$$\begin{aligned} &q^k \int_{v\in\mathbb{V},\frac{\tau(\mathbb{B}(v,v))}{2}\in x(t)^{-1}(1+\mathfrak{p}_{\mathrm{F}}^k)} |y(t)|^{n(\mathbb{V})}\Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))\,\mathrm{d}v \\ &= q^k \int_{x(t)^{-1}(1+\mathfrak{p}_{\mathrm{F}}^k)} \int_{\mathbb{V}_x} |y(t)|^{n(\mathbb{V})}\Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))|\omega_x(v)|\,\mathrm{d}x, \end{aligned}$$

and since Φ is locally constant of compact support, when k is large enough, over $x(t)^{-1}(1+\mathfrak{p}_{\mathrm{F}})^k$, we have

$$\begin{aligned} &\int_{\mathbb{V}_x} \Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))|\omega_x(v)| \\ &= \int_{\mathbb{V}_{x(t)^{-1}}} \Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))|\omega_{x(t)^{-1}}(v)|, \end{aligned}$$

and hence

$$\begin{aligned} &q^k \int_{x(t)^{-1}(1+\mathfrak{p}_{\mathrm{F}}^{k-d})} \int_{\mathbb{V}_x} |y(t)|^{n(\mathbb{V})}\Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))|\omega_x(v)|\,\mathrm{d}x \\ &= |x(t)|^{-1}|y(t)|^{n(\mathbb{V})} \int_{\mathbb{V}_{x(t)^{-1}}} \Phi(z(t,v))\psi^{-1}(\tau(\mathbb{B}_{\mathbb{V}}(v_0,v)))|\omega_{x(t)^{-1}}(v)|. \end{aligned}$$

□

Now consider the $\mathrm{U}(v_0^{\perp})$-invariant morphism

$$\mathrm{X} \to \mathbb{D} : v \mapsto \mathbb{B}(v_0,v).$$

For any $x \in \mathbb{D}$, the preimage of x is

$$\mathrm{X}_x = \{v \in \mathrm{X} \mid \mathbb{B}(v_0, v) = x\} = \left\{ xv_0 + u \mid u \in v_0^\perp, \frac{\tau(\mathbb{B}(u,u))}{2} = 1 - \nu(x) \right\}.$$

Outside $\mathbb{D}^\heartsuit := \{x \in \mathbb{D} \mid \nu(x) = 1\}$, the map $\mathrm{X} \to \mathbb{D}$ is a submersion and by Lemma 2.3.9 again, $\mathrm{U}(v_0^\perp)$ acts transitively on X_x. Write $\mathrm{X}^\heartsuit$ to be the preimage of $\mathbb{D}^\heartsuit$, then we have

$$\mathrm{X}^\heartsuit/\mathrm{U}(v_0^\perp) \cong \mathbb{D}^\heartsuit.$$

The following lemma is well-known for experts, but we also include the proof here for completeness.

Lemma 2.3.11. *(1) When $\mathbb{D} = \mathrm{F}$,*

$$\pi : \mathrm{X} \cong \mathrm{SO}_n/\mathrm{SO}_{n-1} \to \mathbb{A}^1$$

can be identified with the canonical quotient map

$$\mathrm{SO}_n/\mathrm{SO}_{n-1} \to \mathrm{SO}_{n-1} \mathbin{/\mkern-6mu/} \mathrm{SO}_n/\mathrm{SO}_{n-1}.$$

(2) When $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, $\mathrm{X} \to \mathbb{D}$ induces morphisms

$$\begin{tikzcd} \mathrm{GL}_n/\mathrm{GL}_{n-1} \arrow[r] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ \mathrm{PGL}_n/\mathrm{GL}_{n-1} \arrow[r, "\pi"] & \mathbb{A}^1 \end{tikzcd},$$

and π can be identified with the canonical quotient map

$$\mathrm{PGL}_n/\mathrm{GL}_{n-1} \to \mathrm{GL}_{n-1} \mathbin{/\mkern-6mu/} \mathrm{PGL}_n/\mathrm{GL}_{n-1}.$$

(3) When $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$, $\mathrm{X} \to \mathbb{D}$ induces morphisms

$$\begin{tikzcd} \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \arrow[r] \arrow[d] & \mathrm{Mat}_2 \arrow[d, "\det"] \\ \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \arrow[r, "\pi"] & \mathbb{A}^1 \end{tikzcd},$$

and π can be identified with the canonical quotient map

$$\mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \to \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \mathbin{/\mkern-6mu/} \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2.$$

Proof. Write

$$(\mathrm{G},\mathrm{H}) := \begin{cases} (\mathrm{SO}_n, \mathrm{SO}_{n-1}) & \text{if } \mathbb{D} = \mathrm{F}; \\ (\mathrm{PGL}_n, \mathrm{GL}_{n-1}) & \text{if } \mathbb{D} = \mathrm{F} \times \mathrm{F}; \\ (\mathrm{Sp}_{2n}, \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2) & \text{if } \mathbb{D} = \mathrm{Mat}_2(\mathrm{F}). \end{cases}$$

And $\mathrm{X} = \mathrm{G}/\mathrm{H}$ temporarily in this lemma. It has been proved in [Sak21] that we have an isomorphism

$$\mathrm{H} \backslash\!\backslash \mathrm{G}/\mathrm{H} = \mathrm{A}_\mathrm{X} /\!\!/ \mathrm{W}_\mathrm{X},$$

where A_X is the universal Cartan of X and W_X is the little Weyl group, which acts on A_X by inversion. Therefore, in all cases it suffices to show that the restriction of π to the universal Cartan is a generator of $\mathrm{F}[\mathrm{A}_\mathrm{X}]^{\mathrm{W}_\mathrm{X}}$. Write B for the Borel subgrou pf G consisting of upper triangular matrices in the case that $\mathbb{D} = \mathrm{F}$ and $\mathrm{F}\times\mathrm{F}$, and let B be the Borel fixing the complete anisotropic flag of $\{\{e_1\} \subset \{e_1+e_3\} \subset \cdots \subset \{e_1+\cdots+e_{2n-1}\}\}$ in the case $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$.

For (1), we first claim that $\mathrm{B}v_0 \subset \mathrm{X}$ is an open B-orbit. In fact, it is easy to compute that the stabilizers of v_0 in B are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where g is an element in the Borel subgroup of $\mathrm{SO}(n-2)$. Then, when $n = 2k$ is even, we have

$$\dim \mathrm{B}v_0 = k^2 - (k-1)^2 = 2k - 1 = \dim \mathrm{X},$$

and when $n = 2k+1$ is odd,

$$\dim \mathrm{B}v_0 = k(k+1) - (k-1)k = 2k = \dim \mathrm{X}.$$

Then the universal Cartan can be identified with

$$\mathrm{A}_\mathrm{X} = \{\mathrm{diag}\{t, 1, \cdots, 1, t^{-1}\}\},$$

and the restriction of π to $\mathrm{diag}\{t, 1, \cdots, 1, t^{-1}\}v_0$ is exactly $t + t^{-1}$.

For (2), let $x_0 := \begin{pmatrix} 1 & 0 & -1 \\ 0 & \mathrm{I}_{n-2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$, similarly one can easily compute that

$$\dim \mathrm{B}v_0 = 2n-2 = \dim \mathrm{X},$$

then we can identify the universal Cartan as

$$\mathrm{A}_{\mathrm{X}} = x_0^{-1}\{\mathrm{diag}\{t, 1, \cdots, 1, 1\}\}x_0,$$

and the restriction of π to $x_0^{-1}\mathrm{diag}\{t, 1, \cdots, 1, 1\}x_0$ is

$$\begin{aligned}
&\tau\left(\mathbb{B}\left(\left(\begin{pmatrix} -t \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2t} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix}\right), \left(\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix}\right)\right)\right) \\
&= \left(\frac{t}{2} + \frac{1}{2}\right)\left(\frac{1}{2t} + \frac{1}{2}\right) \\
&= \frac{t + t^{-1} + 2}{4}.
\end{aligned}$$

For (3), note that $x_0 = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & \mathrm{I}_{2n-4} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$ is in the open B-orbit since x_0 sending the subspace generated by $\{e_n, f_n\}$ to the space generated by $\{e_1 + e_n, f_1 + f_n\}$, and the stabilizer in B preserving the space generated by $\{e_1 + e_n, f_1 + f_n\}$ has dimension $n^2 + n - (4n-4)$.

If we identify the universal Cartan as $x_0^{-1}\mathrm{diag}\{t, t^{-1}, 1, \cdots, 1\}x_0$, then it is clear that the restriction of π is given by

$$\frac{1}{4}\det\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \frac{t + t^{-1} + 2}{4}.$$

$\square$

According to [Igu00, Chapter 7] again, the measure $|\omega_1|$ on $\mathrm{X} = \mathbb{V}_1$ and the Haar measure on $\mathbb{D}$ give rise to measures $|\mu_x|$ on fibers X_x.

Definition 2.3.12. *Let Φ be a Schwartz function on* X. *We define its orbital integral with respect to the action of $\mathrm{U}(v_0^\perp)$, which is a function on $\mathbb{D}^\heartsuit$, to be*

$$\mathcal{I}(\Phi) : \mathbb{D}^\heartsuit \ni x \mapsto \begin{cases} \int_{\mathrm{X}_{4x-2}} \Phi(v)|\mu_{4x-2}(v)| & \text{for } \mathbb{D} = \mathrm{F} \text{ and } \dim_{\mathrm{F}} \mathbb{V} \equiv 1 \mod 2 \\ \int_{\mathrm{X}_x} \Phi(v)|\mu_x(v)| & \text{otherwise.} \end{cases}$$

We denote this space of orbital integrals by $\mathcal{F}(\mathfrak{X})$.

Remark 2.3.13. *We choose $x = \dfrac{\mathbb{B}(v_0, v)}{4} + \dfrac{1}{2}$, as our coordinates in the case $\mathbb{D} = \mathrm{F}$ and $\dim_{\mathrm{F}} \mathbb{V}$ is odd, to make it compatible with the coordinates chosen in Theorem 2.3.1.*

Proposition 2.3.14. *The morphism $\mathcal{F}(\mathbb{V}) \to \mathrm{Orb}_{\mathbb{V}}(\mathrm{T})$ factors through*

$$\mathcal{F}(\mathbb{V}) \to \mathcal{F}(\mathrm{X}) \to \mathcal{F}(\mathfrak{X}),$$

i.e., we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(\mathbb{V}) \arrow[r, "\mathbb{T}"] \arrow[d, "\mathrm{res}"] & \mathrm{Orb}_{\mathbb{V}}(\mathrm{T}) \\ \mathcal{F}(\mathrm{X}) \arrow[d, "\mathcal{I}"] & \\ \mathcal{F}(\mathfrak{X}) \arrow[uur, "\widetilde{\mathbb{T}}"'] & \end{tikzcd}$$

Moreover, the morphism $\widetilde{\mathbb{T}} : \mathcal{F}(\mathfrak{X}) \to \mathrm{Orb}_{\mathbb{V}}(\mathrm{T})$ is given by the following formulae:

(1) if $\mathbb{D} = \mathrm{F}$ and $\dim_{\mathrm{F}} \mathbb{V}$ is even, we have

$$\widetilde{\mathbb{T}}(\phi) \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} = |\zeta|^{-\frac{\dim_{\mathrm{F}} \mathbb{V}}{2}} \int_{\mathrm{F}^\heartsuit} \phi(x)\psi^{-1}(\zeta^{-1}x)\,\mathrm{d}x;$$

(2) if $\mathbb{D} = \mathrm{F}$ and $\dim_{\mathrm{F}} \mathbb{V}$ is odd, we have

$$\widetilde{\mathbb{T}}(\phi) \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} = \gamma(t, \psi)|\zeta|^{-\frac{\dim_{\mathrm{F}} \mathbb{V}}{2}} \psi^{-1}\left(\frac{-2}{\zeta}\right) \int_{\mathrm{F}^\heartsuit} \phi(x)\psi^{-1}(4\zeta^{-1}x)\,\mathrm{d}x;$$

(3) if $\mathbb{D}=\mathrm{F}\times\mathrm{F}$, *wex have*

$$\widetilde{\mathbb{T}}(\phi)\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)=\left|\frac{t_1}{t_2}\right|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\int_{(\mathrm{F}\times\mathrm{F})^{\heartsuit}}\phi(x_1,x_2)\psi^{-1}(t_2x_1+t_1^{-1}x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.;$$

(4) if $\mathbb{D}=\mathrm{Mat}_2(\mathrm{F})$, *we have*

$$\widetilde{\mathbb{T}}(\phi)\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix}=|\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\int_{\mathrm{Mat}_2(\mathrm{F})^{\heartsuit}}\phi(x)\psi^{-1}\left(\tau\left(x\begin{pmatrix} \xi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right)\mathrm{d}x.$$

Proof. When $\mathbb{D}=\mathrm{F}$ and $\dim_{\mathrm{F}}\mathbb{V}$ is even, this is [GW21, Proposition 11.1]. We also include the proof here for the reader's convenience. Let $\Phi\in\mathcal{F}(\mathbb{V})$, for any integer k, we have

$$\begin{aligned}\mathbb{T}(\Phi)(t)&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\omega_\psi\left(\mathbf{w}t\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}}\omega_\psi\left(t\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(v)\psi^{-1}(\mathbb{B}(v_0,v))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u.\end{aligned}$$

Write $t=\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, we have

$$\begin{aligned}\omega_\psi\left(\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(v)&=|\zeta|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\omega_\psi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(\zeta v)\\&=|\zeta|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\psi\left(\frac{u\cdot\mathbb{B}(\zeta v,\zeta v)}{2}\right)\Phi(\zeta v).\end{aligned}$$

Then we have

$$\begin{aligned}&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\omega_\psi\left(\mathbf{w}t\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}|\zeta|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\int_{\mathbb{V}}\psi\left(\frac{u\cdot\mathbb{B}(\zeta v,\zeta v)}{2}\right)\Phi(\zeta v)\psi^{-1}(\mathbb{B}(v_0,v))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u\\&=|\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\int_{\mathbb{V}}\Phi(v)\psi^{-1}(\mathbb{B}(v_0,\zeta^{-1}v))\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\frac{\mathbb{B}(v,v)}{2}-1\right)\right)\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Note that

$$\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \psi\left(u\left(\frac{(\mathbb{B}(v,v)}{2}-1\right)\right)\mathrm{d}n = \begin{cases} q^k & \text{if } \frac{\mathbb{B}(v,v)}{2} \in 1+\mathfrak{p}_{\mathrm{F}}^k \\ 0 & \text{otherwise.}\end{cases}$$

Then, when k is large enough, we have

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi(\mathbf{w})\omega_\psi(t)\omega_\psi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&= |\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}} \int_{\mathrm{X}} \Phi(v)\psi^{-1}(\mathbb{B}(v_0,\zeta^{-1}v))|\omega_1(v)|\\
&= |\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}} \int_{\mathrm{F}^\heartsuit}\int_{\mathrm{X}_x} \Phi(v)|\mu_x(v)|\psi^{-1}(\zeta^{-1}x)\,\mathrm{d}x\\
&= |\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}} \int_{\mathrm{F}^\heartsuit} \mathcal{I}(\mathrm{res}(\Phi))(x)\psi^{-1}(\zeta^{-1}x)\,\mathrm{d}x.
\end{aligned}$$

In the case that $\dim_{\mathrm{F}}\mathbb{V}$ is odd, using our coordinate, we have

$$\begin{aligned}
\widetilde{\mathbb{T}}(\phi)\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix} &= \gamma(t,\psi)|\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}} \int_{\mathrm{F}^\heartsuit} \mathcal{I}(\mathrm{res}(\Phi))(x)\psi^{-1}(\zeta^{-1}(4x-2))\,\mathrm{d}x\\
&= \gamma(t,\psi)|\zeta|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{2}}\psi^{-1}\left(\frac{-2}{\zeta}\right)\int_{\mathrm{F}^\heartsuit} \mathcal{I}(\mathrm{res}(\Phi))(x)\psi^{-1}(4\zeta^{-1}x)\,\mathrm{d}x.
\end{aligned}$$

When $\mathbb{D}=\mathrm{F}\times\mathrm{F}$, for $\Phi\in\mathcal{F}(\mathbb{V})$ and $t=\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}\in\mathrm{T}$, we have

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&= \int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}} \omega_\psi\left(\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v)\psi^{-1}(\tau(\mathbb{B}(v_0,v)))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u
\end{aligned}$$

Write $\mathbb{V}=\mathrm{F}^n\times\mathrm{F}^n$ and its elements as $(v,v^\vee)$, then we have

$$\omega_\psi\left(\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi((v,v^\vee)) = \left|\frac{t_1}{t_2}\right|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\psi(n\langle t_1v, t_2^{-1}v^\vee\rangle)\Phi((t_1v,t_2^{-1}v^\vee)),$$

hence

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathrm{F}^n\times\mathrm{F}^n}\left|\frac{t_1}{t_2}\right|^{\frac{n}{2}}\psi(u\langle v,v^\vee\rangle)\Phi((v,v^\vee))\cdot\psi^{-1}(\tau(\mathbb{B}(v_0,(t_1^{-1}v,t_2v^\vee))))\,\mathrm{d}v\,\mathrm{d}v^\vee\cdot\psi^{-1}(u)\,\mathrm{d}u\\
&=\left|\frac{t_1}{t_2}\right|^{-\frac{n}{2}}\int_{\mathrm{F}^n\times\mathrm{F}^n}\Phi((v,v^\vee))\psi^{-1}(\langle t_1^{-1}v,e_n^\vee\rangle+\langle e_n,t_2v^\vee\rangle)\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\langle v,v^\vee\rangle-1\right)\right)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}v^\vee
\end{aligned}$$

Since we have

$$\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\langle v,v^\vee\rangle-1\right)\right)\,\mathrm{d}u=\begin{cases} q^k & \text{if } \langle v,v^\vee\rangle\in 1+\mathfrak{p}_{\mathrm{F}}^k\\ 0 & \text{otherwise,}\end{cases}$$

hence when k is large enough,

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\left|\frac{t_1}{t_2}\right|^{-\frac{n}{2}}\int_{\mathrm{X}}\Phi((v,v^\vee))\psi^{-1}(\langle t_1^{-1}v,e_n^\vee\rangle+\langle e_n,t_2v^\vee\rangle)|\omega_1((v,v^\vee))|\\
&=\left|\frac{t_1}{t_2}\right|^{-\frac{n}{2}}\int_{(\mathrm{F}\times\mathrm{F})^\heartsuit}\mathcal{I}(\mathrm{res}(\Phi))(x_1,x_2)\psi^{-1}(t_2x_1+t_1^{-1}x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

If $\mathbb{D}=\mathrm{Mat}_2(\mathrm{F})$, for $\Phi\in\mathcal{F}(\mathbb{V})$ and $t=\begin{pmatrix}\xi & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \xi^{-1}\end{pmatrix}\in\mathrm{T}$, we have

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}}\omega_{\psi,\mathbb{X},\mathbb{Y}}\left(\begin{pmatrix}\xi & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \xi^{-1}\end{pmatrix}n\right)\Phi(v)\psi^{-1}(\tau(\mathbb{B}(v_0,v)))\,\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u
\end{aligned}$$

Since we have

$$\omega_\psi\left(tu\right)\Phi(v) = |\xi|^{\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\psi\left(\frac{u\cdot\tau\left(\mathbb{B}_{\mathbb{V}}\left(v\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}, v\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right)\right)}{2}\right)\Phi\left(v\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\right),$$

then

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathbb{V}}|\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\psi\left(\frac{u\cdot\tau\left(\mathbb{B}_{\mathbb{V}}(v,v)\right)}{2}\right)\Phi(v)\cdot\psi^{-1}\left(\tau\left(\mathbb{B}\left(v_0, v\begin{pmatrix}\xi^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)\right)\mathrm{d}v\cdot\psi^{-1}(u)\,\mathrm{d}u\\
&=|\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\int_{\mathbb{V}}\Phi(v)\psi^{-1}\left(\tau\left(\mathbb{B}_{\mathbb{V}}(v_0,v)\begin{pmatrix}\xi^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\frac{\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}-1\right)\right)\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

Again

$$\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\psi\left(u\left(\frac{\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}-1\right)\right)\mathrm{d}u = \begin{cases} q^k & \text{if } \frac{\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\in 1+\mathfrak{p}_{\mathrm{F}}^k\\ 0 & \text{otherwise,}\end{cases}$$

when k is large enough,

$$\begin{aligned}
&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\omega_\psi\left(\mathbf{w}t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=|\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\int_{\mathrm{X}}\Phi(v))\psi^{-1}\left(\tau\left(\mathbb{B}_{\mathbb{V}}(v_0,v)\begin{pmatrix}\xi^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)|\omega_1(v)|\\
&=|\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}}\int_{\mathrm{Mat}_2(\mathrm{F})^{\heartsuit}}\mathcal{I}(\mathrm{res}(\Phi))(x)\psi^{-1}\left(\tau\left(x\begin{pmatrix}\xi^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)\mathrm{d}x.
\end{aligned}$$

□

To facilitate a comparison with the transfer operators established in Theorem 2.3.1, we examine each case in detail. We begin with the case $\mathbb{D} = \mathrm{F}$, which is the most direct. Let $\mathcal{S}(\mathfrak{X}) = \mathcal{F}(\mathfrak{X})\,\mathrm{d}x$ and $\mathcal{S}_{\mathbb{V}}(\mathrm{T}) := \mathrm{Orb}_{\mathbb{V}}(\mathrm{T})|\zeta^2|\,\mathrm{d}^\times\zeta$ denote the respective spaces of measures. Retaining the notation $\overline{\mathbb{T}}$ for the extension to $\mathcal{S}(\mathfrak{X})$.

Proposition 2.3.15. *In the case where* $\dim\mathbb{V}$ *is even, let* $\mathcal{T}$ *denote the transfer established in Theorem 2.3.1. Then, on the space of push-forward measures, we have the identity*

$$\widetilde{\mathbb{T}}\circ\mathcal{T} = \mathrm{Id}.$$

Proof. According to Proposition 2.3.14, for $\phi \in \mathcal{F}(\mathfrak{X})$, we have

$$\widetilde{\mathbb{T}}(\phi)(\zeta) = |\zeta|^{-\frac{\dim_{\mathbb{D}} \mathbb{V}}{2}} \int_{\mathbb{D}^{\heartsuit}} \phi(x)\psi^{-1}(\zeta^{-1}x)\, \mathrm{d}x = (|\cdot|^{-n} \cdot \iota \circ \mathbb{F}_{\psi}(\phi))(\zeta). \tag{2.7}$$

□

Proposition 2.3.16. *When* $\dim_{\mathrm{F}} \mathbb{V}$ *is odd, we have*

$$\widetilde{\mathbb{T}} \circ \mathcal{T} = \gamma(\cdot, \psi)\psi\left(\frac{2}{\cdot}\right) \cdot \left(|\cdot|^{\frac{1}{2}} \circ \circ \mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot|\right)\left(\frac{\cdot}{4}\right).$$

Proof. The result is also a direct consequence of Proposition 2.3.14. □

Next, consider the case that $\mathbb{D} = \mathrm{F} \times \mathrm{F}$. Consider the following $\mathbb{G}_m$-action on $\mathbb{A}^2$:

$$z(x_1, x_2) := (zx_1, z^{-1}x_2),\ \forall z \in \mathbb{G}_m,\ (x_1, x_2) \in \mathbb{A}^2.$$

It is clear that $\mathbb{A}^2 /\!\!/ \mathbb{G}_m = \mathbb{A}^1$ via the map $(x_1, x_2) \mapsto x_1x_2$.

Lemma 2.3.17. *For any* $\phi \in \mathcal{F}(\mathfrak{X})$, $x_1, x_2 \in \mathrm{F}^{\times}$ *such that* $x_1x_2 \neq 0, 1$,

$$\int_{\mathbb{G}_m} \phi(z^{-1}x_1, zx_2)\, \mathrm{d}z$$

is absolutely converget.

Proof. Let $\Phi \in \mathcal{F}(\mathrm{X})$ and $\phi = \mathcal{I}(\Phi)$. Note that for fixed $(x_1, x_2) \in (\mathrm{F} \times \mathrm{F})^{\heartsuit}$ such that $x_1x_2 \neq 0, 1$,

$$\phi(z^{-1}x_1, zx_2) \neq 0$$

only when the support of Φ has a non-empty intersection with $\mathrm{X}_{(z^{-1}x_1, zx_2)}$. Since the support of Φ is compact, and $\mathrm{X} \to \mathrm{F}$ is a submersion away from $0, 1$, the integral

$$\int_{\mathbb{G}_m} \phi(z^{-1}x_1, zx_2)\, \mathrm{d}z$$

is actually over some compact subsect of $\mathrm{F}^{\times}$, hence is absolutely convergent. □

On the other hand, consider the center integrals of elements in $\mathrm{Orb}_{\mathbb{V}}(\mathrm{T})$, we also have

Lemma 2.3.18. *For $f \in \mathrm{Orb}_{\mathbb{V}}(\mathrm{T})$,*

$$\int_{\mathbb{G}_m} f\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \mathrm{d}z$$

is absolutely convergent for $\xi \in \mathrm{F}^\times$.

Proof. According to the proof of Proposition 2.3.14, the choice of k in the regularized integral is independent of t and only depends on the function Φ. Let us fix such a k. We first claim that when $|z|$ is large enough,

$$f\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) = 0.$$

In fact, we may assume $f = \widetilde{\mathbb{T}}(\phi)$ for some $\phi = \mathcal{I}(\mathrm{res}(\Phi))$, and $\Phi = \Phi_1 \otimes \cdots \Phi_n \otimes \Psi_1 \otimes \Psi_n$ is a pure tensor. Then we have

$$\begin{aligned}
&f\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \\
&= \int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \int_{\mathrm{F}^n \times \mathrm{F}^n} |\xi|^{-\frac{\dim_{\mathbb{D}} \mathbb{V}}{2}} \psi(u(x_1 y_1 + \cdots x_n y_n)) \Phi_1(x_1) \cdots \Phi_n(x_n) \Psi_1(y_1) \cdots \Psi_n(y_n) \\
&\cdot \psi^{-1}(\xi^{-1} z^{-1} x_n + z y_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{d}y_1 \cdots \mathrm{d}y_n \cdot \psi^{-1}(u) \, \mathrm{d}u \\
&= \int_{\mathfrak{p}_{\mathrm{F}}^{-k}} \int_{\mathrm{F}^n} |\xi|^{-\frac{\dim_{\mathbb{D}} \mathbb{V}}{2}} \mathbb{F}_{\psi^{-1}}(\Phi_1)(u y_1) \cdots \mathbb{F}_{\psi^{-1}}(\Phi_{n-1})(u y_{n-1}) \mathbb{F}_{\psi^{-1}}(\Phi_n)(-\xi^{-1} z^{-1} + u y_n) \\
&\cdot \Psi_1(y_1) \cdots \Psi_n(y_n) \psi^{-1}(z y_n) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n \psi^{-1}(u) \, \mathrm{d}u.
\end{aligned}$$

For fixed ξ, when z is large enough, we have $\mathbb{F}_{\psi^{-1}}(\Phi_n)(\xi^{-1} z^{-1} - u y_n) = \mathbb{F}_{\psi^{-1}}(\Phi_n)(u y_n)$, and

$$\begin{aligned}
&\int_{\mathrm{F}} \mathbb{F}_{\psi^{-1}}(\Phi_n)(-\xi^{-1} z^{-1} + u y_n) \Psi_n(y_n) \psi^{-1}(z y_n) \, \mathrm{d}y_n \\
&= \int_{\mathrm{F}} \mathbb{F}_{\psi^{-1}}(\Phi_n)(u y_n) \Psi_n(y_n) \psi^{-1}(z y_n) \, \mathrm{d}y_n.
\end{aligned}$$

It is not hard to see that when $|u| \leq q^k$,

$$\int_{\mathrm{F}} \mathbb{F}_{\psi^{-1}}(\Phi_n)(u y_n) \Psi_n(y_n) \psi^{-1}(z y_n) \, \mathrm{d}y_n = 0$$

when $|z|$ is large enough. Similarly by changing z to z^{-1}, we see

$$f\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right) = 0.$$

when $|z|$ is small enough, hence

$$\int_{\mathbb{G}_m} f\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right) \mathrm{d}z$$

is absolutely convergent for $\xi \in \mathrm{F}^\times$.

□

Definition 2.3.19. *(1) For any $\phi \in \mathcal{F}(\mathfrak{X})$, and $x \in \mathrm{F} \setminus \{0,1\}$, write*

$$\widetilde{\phi}(x) := \int_{\mathrm{F}} \phi(z^{-1}x, z)\, \mathrm{d}^\times z.$$

We will use $\mathcal{F}(\overline{\mathfrak{X}})$ to denote the space of such functions and $\mathcal{S}(\overline{\mathfrak{X}}) := \mathcal{F}(\overline{\mathfrak{X}})\, \mathrm{d}x$.

(2) Let $\mathrm{Orb}_{\mathbb{V}}(\overline{\mathrm{T}})$ be the space of functions obtained from $\mathrm{Orb}(\mathrm{T})$ by taking integrals over the center, namely, they are functions of $\xi \in \mathrm{F}^\times$ of the form

$$\widetilde{f}(\xi) := \int_{\mathbb{G}_m} f\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right) \mathrm{d}z, \ f \in \mathrm{Orb}_{\mathbb{V}}(\mathrm{T}).$$

Write $\mathcal{S}_{\mathbb{V}}(\overline{\mathrm{T}}) := \mathrm{Orb}_{\mathbb{V}}(\overline{\mathrm{T}})|\xi|\, \mathrm{d}^\times \xi$.

Lemma 2.3.20. *We have a commutative diagram*

$$\begin{tikzcd} \mathcal{F}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{Orb}_{\mathbb{V}}(\mathrm{T}) \arrow[d] \\ \mathcal{S}(\overline{\mathfrak{X}}) \arrow[r] & \mathcal{S}_{\mathbb{V}}(\overline{\mathrm{T}}), \end{tikzcd}$$

where the bottom arrow is given by

$$\overline{\mathbb{T}} : |\cdot|^{-\frac{n}{2}} \circ \iota \circ \mathbb{F}_\psi \circ |\cdot|^{-1} \circ \iota \circ \mathbb{F}_\psi$$

in the sense of measures.

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\mathrm{F}^\times)$ be a test function, and $f = \widetilde{\mathbb{T}}(\phi)$ for $\phi \in \mathcal{F}(\mathfrak{X})$, then we have

$$\langle \widetilde{f}, \varphi \rangle = \int_{\mathrm{F}} \widetilde{f}(\xi)\varphi(\xi)\,\mathrm{d}\xi.$$

Using Fubini's theorem, we have

$$\begin{aligned}\langle \widetilde{f}, \varphi \rangle &= \int_{\mathrm{F}} \int_{\mathrm{F}^\times} |\xi|^{-\frac{n}{2}} \int_{(\mathrm{F}\times\mathrm{F})^\heartsuit} \phi(\xi_1, \xi_2)\psi^{-1}(z\xi_1 + \xi^{-1}z^{-1}\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \frac{\mathrm{d}z}{|z|}\varphi(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}^\times} \int_{(\mathrm{F}\times\mathrm{F})^\heartsuit} \phi(\xi_1, \xi_2)\psi^{-1}(z\xi_1 + \xi^{-1}z^{-1}\xi_2)\varphi(\xi)|\xi|^{-\frac{n}{2}}\,\mathrm{d}\xi\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \frac{\mathrm{d}z}{|z|} \\ &= \int_{\mathrm{F}^\times} \int_{(\mathrm{F}\times\mathrm{F})^\heartsuit} \phi(\xi_1, \xi_2)\psi^{-1}(z\xi_1)\mathbb{F}_\psi(|\cdot|^{-2} \circ \iota \circ |\cdot|^{-\frac{n}{2}}\varphi)(z^{-1}\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \frac{\mathrm{d}z}{|z|}.\end{aligned}$$

As $\mathbb{F}_\psi(|\cdot|^{-2} \circ \iota \circ |\cdot|^{-\frac{n}{2}}\varphi)$ vanishes near 0 and ∞, we can apply the Fubini's theorem again to obtain

$$\begin{aligned}\langle \widetilde{f}, \varphi \rangle &= \int_{(\mathrm{F}\times\mathrm{F})^\heartsuit} \phi(\xi_1, \xi_2) \int_{\mathrm{F}^\times} \mathbb{F}_\psi(|\cdot|^{-2} \circ \iota \circ |\cdot|^{-\frac{n}{2}}\varphi)(z^{-1})|z^{-1}|^{-1}\psi^{-1}(\xi_1\xi_2 z)\frac{\mathrm{d}z}{|z|^2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &= \langle \widetilde{\phi}, \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-1} \circ \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-\frac{n}{2}}\varphi \rangle.\end{aligned}$$

Note that for a measure μ and a test function φ, we have

$$\langle \iota(\mu), \varphi \rangle = \langle \mu, |\cdot|^{-2} \circ \iota(\varphi) \rangle,$$

hence

$$\begin{aligned}&\langle \widetilde{\phi}, \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-1} \circ \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-\frac{n}{2}}\varphi \rangle \\ &= \langle |\cdot|^{-\frac{n}{2}} \circ \iota \circ \mathbb{F}_\psi \circ |\cdot|^{-1} \circ \iota \circ \mathbb{F}_\psi, \varphi \rangle.\end{aligned}$$

□

Proposition 2.3.21. *Let $\mathcal{T}$ be the transfer operator in Theorem 2.3.1, then we have*

$$\overline{\mathbb{T}} \circ \mathcal{T} = \mathrm{Id}.$$

Proof. This is a direct consequence of Lemma 2.3.20. □

The last case is when $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$. Consider the action of SL_2 on Mat_2 by left or right multiplication. Then according to [GW09, Theorem 5.3.3], $\mathrm{Mat}_2 /\!\!/ \mathrm{SL}_2 = \mathbb{A}^1$ via $g \mapsto \det g$. Write $\mathcal{S}(\mathfrak{X}) := \mathcal{F}(\mathfrak{X})\,\mathrm{d}x$ for measures on $\mathrm{Mat}_2(\mathrm{F})^\heartsuit$ by multiplying an additive Haar measure.

Lemma 2.3.22. *For $\phi \in \mathcal{F}(\mathfrak{X})$, and $x \in \mathrm{F} \setminus \{0,1\}$,*

$$\widetilde{\phi}(x) := \int_{\mathrm{SL}_2} \phi\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}g$$

is absolutely convergent.

Proof. This is similar to Lemma 2.3.17. □

In this case, we have to make some modifications to the transfer operator $\widetilde{\mathbb{T}}$. Now, consider the embedding

$$\mathrm{SL}_2 \hookrightarrow \mathrm{SO}_4 : g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Consider the adjoint action of PGL_2 on $\mathfrak{gl}_2 = \mathrm{Mat}_2(\mathrm{F})$, which is equipped with the quadratic form given by the determinant. The ordered basis

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

gives an embedding $\mathrm{PGL}_2 \to \mathrm{O}_4$ such that

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \mapsto \begin{pmatrix} t & & & \\ & 1 & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & n & \\ & 1 & & n \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \mapsto \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

We also identify PGL_2 with its image in O_4. Note that PGL_2 normalizes SL_2. Then, let us consider the function

$$\begin{aligned} \overline{\Omega}(\Phi) : \mathrm{PGL}_2 &\to \mathbb{C} \\ g &\mapsto \int_{\mathrm{SL}_2} \Omega_\psi(\Phi)(gg')\, \mathrm{d}g'. \end{aligned}$$

For $\Phi \in \mathcal{F}(\mathbb{V})$, replace Φ by $\omega_\psi(g)\Phi$ for $g \in \mathrm{SL}_2$ in Proposition 2.3.14,we have

$$\begin{aligned}
&\int_{\mathrm{N}}^{*} \omega_\psi(\mathbf{w})\omega_\psi\left(\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix}\right)\omega_\psi(u)\omega_\psi(g)\Phi(v_0)\psi^{-1}(u)\,\mathrm{d}u \\
&= |\xi|^{-\frac{\dim_{\mathrm{F}}\mathbb{V}}{4}} \int_{\mathrm{Mat}_2(\mathrm{F})^{\heartsuit}} \mathcal{I}(\mathrm{res}(\Phi))(x)\psi^{-1}\left(\tau\left(xg^{-1}\begin{pmatrix} \xi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right)\mathrm{d}x \\
&=: \widehat{\mathbb{T}}(\phi)(\xi, g),
\end{aligned}$$

where $\phi = \mathcal{I}(\mathrm{Res}(\Phi))$. Denote the space of such functions by $\mathrm{Orb}_{\mathbb{V}}(\mathrm{T} \times \mathrm{SL}_2)$.

Lemma 2.3.23. *For $\phi \in \mathcal{F}(\mathfrak{X})$,*

$$\int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\phi)(\xi, g)\,\mathrm{d}g$$

is absolutely convergent for $\xi \in \mathrm{F}^\times$.

Proof. Let $\phi = \mathcal{I}(\mathrm{res}(\Phi))$ with $\Phi \in \mathcal{F}(\mathbb{V})$. Assume $\mathrm{K}' \subset \mathrm{K} = \mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})$ such that $\omega_\psi(k)\Phi = \Phi$ for any $k \in \mathrm{K}'$. According to the Iwasawa decomposition, we have

$$\begin{aligned}
\int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\phi)(\xi, g)\,\mathrm{d}g &= \int_{\mathrm{F}\times\mathrm{F}^\times}\int_{\mathrm{K}} \widehat{\mathbb{T}}(\phi)\left(\xi, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} k\right)|m|^{-2}\,\mathrm{d}x\frac{\mathrm{d}m}{|m|}\,\mathrm{d}k \\
= \sum_{k_i \in \mathrm{K}/\mathrm{K}'} \mathrm{vol}(\mathrm{K}') &\int_{\mathrm{F}\times\mathrm{F}^\times} \widehat{\mathbb{T}}(\phi_i)\left(\xi, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}\right)|m|^{-2}\,\mathrm{d}x\frac{\mathrm{d}m}{|m|},
\end{aligned}$$

where $\phi_i = \mathcal{I}(\mathrm{res}(\omega_\psi(k_i)\Phi))$. For simplicity of notations, replace Φ by $\omega_\psi(k_i)\Phi$, it suffices to show that

$$\int_{\mathrm{F}\times\mathrm{F}^\times} \widehat{\mathbb{T}}(\phi)\left(\xi, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}\right)|m|^{-2}\,\mathrm{d}x\frac{\mathrm{d}m}{|m|}$$

is absolutely convergent.

Note that we have

$$\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\xi&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\xi^{-1}\end{pmatrix}\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&x&0&0\\0&1&0&0\\0&0&1&x\\0&0&0&1\end{pmatrix}\begin{pmatrix}m&0&0&0\\0&m^{-1}&0&0\\0&0&m&0\\0&0&0&m^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}1&\xi x&0&0\\0&1&0&0\\0&0&1&\xi x\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\xi m&0&0&0\\0&m^{-1}&0&0\\0&0&m&0\\0&0&0&\xi^{-1}m^{-1}\end{pmatrix}\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

then

$$\omega_\psi\left(\mathbf{w}\begin{pmatrix}\xi&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\xi^{-1}\end{pmatrix}\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&x&0&0\\0&1&0&0\\0&0&1&x\\0&0&0&1\end{pmatrix}\begin{pmatrix}m&0&0&0\\0&m^{-1}&0&0\\0&0&m&0\\0&0&0&m^{-1}\end{pmatrix}\right)\Phi(v_0)$$

$$=\omega_\psi\left(\mathbf{w}\begin{pmatrix}\xi m&0&0&0\\0&m^{-1}&0&0\\0&0&m&0\\0&0&0&\xi^{-1}m^{-1}\end{pmatrix}\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\Phi\left(v_0\begin{pmatrix}1&\xi x\\0&1\end{pmatrix}\right)$$

$$=\int_{\mathbb{V}}\omega_\psi\left(\begin{pmatrix}\xi m&0&0&0\\0&m^{-1}&0&0\\0&0&m&0\\0&0&0&\xi^{-1}m^{-1}\end{pmatrix}\begin{pmatrix}1&0&u&0\\0&1&0&u\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\Phi(v)$$

$$\cdot\,\psi^{-1}\left(\tau\left(\mathbb{B}_{\mathbb{V}}\left(v_0\begin{pmatrix}1&\xi x\\0&1\end{pmatrix},v\right)\right)\right)\,\mathrm{d}v$$

$$=\int_{\mathbb{V}}|\xi m\cdot m^{-1}|^n\psi\left(\frac{u\cdot\mathbb{B}_{\mathbb{V}}\left(v\begin{pmatrix}\xi m&0\\0&m^{-1}\end{pmatrix},v\begin{pmatrix}\xi m&0\\0&m^{-1}\end{pmatrix}\right)}{2}\right)\Phi\left(v\begin{pmatrix}\xi m&0\\0&m^{-1}\end{pmatrix}\right)$$

$$\psi^{-1}\left(\tau\left(\mathbb{B}_{\mathbb{V}}\left(v_0\begin{pmatrix}1&\xi x\\0&1\end{pmatrix},v\right)\right)\right)\,\mathrm{d}v$$

$$=\int_{\mathbb{V}}|\xi|^{-n}\psi\left(\frac{u\cdot\mathbb{B}_{\mathbb{V}}(v,v)}{2}\right)\Phi(v)\psi^{-1}\left(\tau\left(\mathbb{B}_{\mathbb{V}}\left(v_0\begin{pmatrix}1&\xi x\\0&1\end{pmatrix},v\begin{pmatrix}\xi^{-1}m^{-1}&0\\0&m\end{pmatrix}\right)\right)\right)\,\mathrm{d}v.$$

Write $v = \left(\begin{pmatrix} x_1 & z_1 \\ w_1 & y_1 \end{pmatrix}, \cdots, \begin{pmatrix} x_n & z_n \\ w_n & y_n \end{pmatrix}\right)$, then

$$\begin{aligned}
\tau\left(\mathbb{B}_{\mathbb{V}}\left(v_0\begin{pmatrix} 1 & \xi x \\ 0 & 1 \end{pmatrix}, v\begin{pmatrix} \xi^{-1}m^{-1} & 0 \\ 0 & m \end{pmatrix}\right)\right) &= \tau\left(\begin{pmatrix} 1 & -\xi x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_n & z_n \\ w_n & y_n \end{pmatrix}\begin{pmatrix} \xi^{-1}m^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}\right) \\
&= \tau\left(\begin{pmatrix} \xi^{-1}m^{-1}x_n - xm^{-1}w_n & * \\ * & my_n \end{pmatrix}\right) \\
&= \xi^{-1}m^{-1}x_n + my_n - m^{-1}xw_n,
\end{aligned}$$

so

$$\begin{aligned}
&\omega_\psi\left(\mathbf{w}\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m^{-1} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m^{-1} \end{pmatrix}\right)\Phi(v_0) \\
&= |\xi|^{-n}\int_{\mathrm{F}^{4n}}\Phi\left(\left(\begin{pmatrix} x_1 & z_1 \\ w_1 & y_1 \end{pmatrix}, \cdots, \begin{pmatrix} x_n & z_n \\ w_n & y_n \end{pmatrix}\right)\right) \\
&\cdot \psi(u(x_1y_1 + \cdots + x_ny_n - z_1w_1 - \cdots - z_nw_n))\psi^{-1}(\xi^{-1}m^{-1}x_n + my_n - m^{-1}xw_n).
\end{aligned}$$

Without loss of generality, we may assume

$$\Phi = \Phi_1(x_1, \cdots, x_n, y_1, \cdots, y_n)\Phi_2(z_1, \cdots, z_n, w_1, \cdots, w_n)$$

with $\Phi_1, \Phi_2 \in \mathcal{F}(\mathrm{F}^n)$. Then for some k large enough,

$$\begin{aligned}
&\widehat{\mathbb{T}}(\phi)\left(\xi, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}\right) \\
&= |\xi|^{-n}\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathrm{F}^{2n}}\Phi_1(x_1, \cdots, x_n, y_1, \cdots, y_n)\psi(u(x_1y_1 + \cdots + x_ny_n))\psi^{-1}(\xi^{-1}m^{-1}x_n + my_n) \\
&\cdot \int_{\mathrm{F}^{2n}}\Phi_2(z_1, \cdots, z_n, w_1, \cdots, w_n)\psi(-u(z_1w_1 + \cdots + z_nw_n))\psi^{-1}(-m^{-1}xw_m)\psi^{-1}(u)\,\mathrm{d}u.
\end{aligned}$$

From the proof of Lemma 2.3.18,

$$\int_{\mathrm{F}^{2n}}\Phi_1(x_1, \cdots, x_n, y_1, \cdots, y_n)\psi(u(x_1y_1 + \cdots + x_ny_n))\psi^{-1}(\xi^{-1}m^{-1}x_n + my_n)\,\mathrm{d}x_1\cdots\mathrm{d}y_n \neq 0$$

implies that $M_1 \leq |m| \leq M_2$ for some $M_1, M_2 > 0$. Then since

$$\begin{aligned}
&\int_{\mathrm{F}^{2n}}\Phi_2(z_1, \cdots, z_n, w_1, \cdots, w_n)\psi^{-1}(-u(z_1w_1 + \cdots + z_nw_n))\psi^{-1}(-m^{-1}xw_n)\,\mathrm{d}z_1\cdots\mathrm{d}w_n \\
&= \int_{\mathrm{F}^n}\mathbb{F}_{\psi,1}\circ\cdots\circ\mathbb{F}_{\psi,n}\Phi_2(-uw_1, \cdots, -uw_{n-1}, -uw_n, w_1, \cdots, w_n)\psi^{-1}(-m^{-1}xw_n)\,\mathrm{d}w_1\cdots\mathrm{d}w_n,
\end{aligned}$$

which is non-zero only when $|x| \leq A$ for some $A > 0$ depending on Φ_2, $\mathfrak{p}_{\mathrm{F}}^{-k}$, and M. Therefore,

$$\int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\phi)(\xi, g)\,\mathrm{d}g$$

is absolutely convergent for $\xi \in \mathrm{F}^\times$.

$\square$

Definition 2.3.24. *(1) For $\phi \in \mathcal{S}(\mathfrak{X})$, let $\widetilde{\phi}$ be the push-forward of the measure ϕ to $\mathbb{A}^1$ along the determinant map. We will use $\mathcal{S}(\overline{\mathfrak{X}})$ to denote the space of such measures.*

(2) Let $\mathrm{Orb}_{\mathbb{V}}(\overline{\mathrm{T}})$ be the space of functions

$$\xi \mapsto \int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\phi)(\xi, g)\,\mathrm{d}g,$$

and $\mathcal{S}_{\mathbb{V}}(\overline{\mathrm{T}}) := \mathrm{Orb}_{\mathbb{V}}(\overline{\mathrm{T}})|\xi|\,\mathrm{d}^\times\xi$.

Remark 2.3.25. *Observe that*

$$\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}\begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

with $\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \in \mathrm{SL}_2(\mathrm{F})$*, hence on the maximal torus of* PGL_2*,*

$$\int_{\mathrm{N}}^{*} \overline{\Omega}(\Phi)\left(\mathbf{w}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} n\right)\psi^{-1}(n)\,\mathrm{d}n = \int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\mathcal{I}(\mathrm{Res}(\Phi)))(\xi, g)\,\mathrm{d}g.$$

Lemma 2.3.26. *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{F}(\mathfrak{X}) & \longrightarrow & \mathrm{Orb}_{\mathbb{V}}(\mathrm{T} \times \mathrm{SL}_2) \\ \downarrow & & \downarrow \\ \mathcal{S}(\overline{\mathfrak{X}}) & \longrightarrow & \mathcal{S}_{\mathbb{V}}(\overline{\mathrm{T}}), \end{array}$$

where the bottom arrow is given by

$$\overline{\mathbb{T}} : |\cdot|^{-n+1} \circ \iota \circ \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ \mathbb{F}_\psi$$

in the sense of measures or distributions.

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\mathrm{F}^\times)$ be a test function, and $f = \widehat{\mathbb{T}}(\phi)$ with $\phi \in \mathcal{F}(\mathfrak{X})$, then we have

$$\langle \widetilde{f}, \varphi \rangle = \int_{\mathrm{F}} |\xi|^{-n} \int_{\mathrm{SL}_2} \int_{\mathrm{Mat}_2(\mathrm{F})^\heartsuit} \phi(x)\psi^{-1}\left(\mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & \zeta^{-1} \end{pmatrix} gx\right)\right) \mathrm{d}x \, \mathrm{d}g \cdot \varphi(\xi) \, \mathrm{d}\xi.$$

Write $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and using the Brahut decomposition to write

$$g = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \tau\left(xg^{-1}\begin{pmatrix} \xi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) &= \mathrm{tr}\left(\begin{pmatrix} \xi^{-1}tuvx_1 + \xi^{-1}t^{-1}x_1 - \xi^{-1}tux_2 & * \\ * & -tvx_3 + tx_4 \end{pmatrix}\right) \\ &= \xi^{-1}tuvx_1 + \xi^{-1}t^{-1}x_1 - \xi^{-1}tux_2 - tvx_3 + tx_4 \end{aligned}$$

As φ is a Schwartz function, we can apply Fubini's theorem, then

$$\begin{aligned} \langle \widetilde{f}, \varphi \rangle &= \int_{\mathrm{F}^\times \times \mathrm{F} \times \mathrm{F}} \int_{\mathrm{Mat}_2(\mathrm{F})^\heartsuit} \phi(x)\psi^{-1}(tx_4 - tvx_3) \\ &\cdot \int_{\mathrm{F}} \varphi(\zeta)|\zeta|^{-n}\psi^{-1}(\xi^{-1}tuvx_1 + \xi^{-1}t^{-1}x_1 - \xi^{-1}tux_2) \, \mathrm{d}\xi \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4 |t|^2 \frac{\mathrm{d}t}{|t|} \, \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

Since

$$\begin{aligned} &\int_{\mathrm{F}} \varphi(\xi)|\xi|^{-n}\psi^{-1}(\xi^{-1}tuvx_1 + \xi^{-1}t^{-1}x_1 - \xi^{-1}tux_2) \, \mathrm{d}\zeta \\ &= (\mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)(tuvx_1 + t^{-1}x_1 - tux_2) \end{aligned}$$

and $(\mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)$ also vanishes at 0 and ∞, we can first integral over v and replace v by $\dfrac{v + tux_2 - t^{-1}x_1}{tux_1}$, then

$$\begin{aligned}
&\int_{\mathrm{F}} (\mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)(tuvx_1 + t^{-1}x_1 - tux_2)\psi^{-1}(-tvx_3)\,\mathrm{d}v \\
&= \int_{\mathrm{F}} (\mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)(v)\psi^{-1}\left(-v\frac{x_3}{ux_1}\right)\frac{1}{|tux_1|}\,\mathrm{d}v \cdot \psi^{-1}\left(-\frac{-t^{-1}x_1x_3 + tux_2x_3}{ux_1}\right) \\
&= (|\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\left(\frac{x_3}{ux_1}\right)\frac{1}{|tux_1|}\psi^{-1}\left(-\frac{-t^{-1}x_1x_3 + tux_2x_3}{ux_1}\right).
\end{aligned}$$

We can further integral over u first, and make a change of variable by replacing u by $\dfrac{x_3}{ux_1}$, then

$$\begin{aligned}
&\int_{\mathrm{F}} (|\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\left(\frac{x_3}{ux_1}\right)\frac{1}{|u|}\psi^{-1}\left(\frac{t^{-1}x_3}{u}\right)\,\mathrm{d}u \\
&= \int_{\mathrm{F}} (|\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)(u)\psi^{-1}(t^{-1}x_1u)\frac{\mathrm{d}u}{|u|} \\
&= (\mathbb{F}_\psi \circ |\cdot|^{-1} \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\left(\frac{x_1}{t}\right).
\end{aligned}$$

Finally, replace t by tx_1, then we have

$$\begin{aligned}
&\int_{\mathrm{F}} (\mathbb{F}_\psi \circ |\cdot|^{-1} \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\left(\frac{x_1}{t}\right)\frac{1}{|tx_1|}\psi^{-1}\left(t\frac{x_1x_4 - x_2x_3}{x_1}\right)|t|^2\frac{\mathrm{d}t}{|t|} \\
&= \int_{\mathrm{F}} (\mathbb{F}_\psi \circ |\cdot|^{-1} \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\left(\frac{1}{t}\right)\psi^{-1}(t(x_1x_4 - x_2x_3))\,\mathrm{d}t \\
&= (\mathbb{F}_\psi \circ \iota \circ \mathbb{F}_{\psi^{-1}} \circ |\cdot|^{-1} \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)(x_1x_4 - x_2x_3),
\end{aligned}$$

hence

$$\begin{aligned}
\langle \widetilde{f}, \varphi\rangle &= \langle \widetilde{\phi}, (\mathbb{F}_\psi \circ \iota \circ \mathbb{F}_\psi \circ |\cdot|^{-1} \circ |\cdot|^{-2} \circ \iota \circ |\cdot|^{-n})(\varphi)\rangle \\
&= \langle (|\cdot|^{-n} \circ \iota \circ |\cdot|^{-1} \circ \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ \mathbb{F}_\psi)(\phi), \varphi\rangle \\
&= \langle (|\cdot|^{-n+1} \circ \iota \circ \mathbb{F}_\psi \circ |\cdot|^{-2} \circ \iota \circ \mathbb{F}_\psi)(\phi), \varphi\rangle.
\end{aligned}$$

□

Proposition 2.3.27. *Let $\mathcal{T}$ be the transfer operator in Theorem 2.3.1, then we have*

$$\overline{\mathbb{T}} \circ \mathcal{T} = \mathrm{Id}.$$

Proof. It follows from Lemma 2.3.26. □

2.4 Fundamental lemma of type D_n

The results of this section are fundamentally contained in the works of S. Rallis [Ral82] and W. Gan-X. Lei [GW21]. While we provide a reformulation of these results in the present framework, the author wishes to emphasize that the credit for the fundamental lemma for the full Hecke algebra in the D_n case belongs to S. Rallis. We include the details here to ensure the manuscript remains self-contained and to clarify the compatibility with our notation.

In this section, let $\mathbb{D} = \mathrm{F}$, and $\dim_{\mathrm{F}} \mathbb{V}$ is even such that there is an ordered basis

$$\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$$

satisfying

$$\mathbb{B}_{\mathbb{V}}(e_i, e_j) = \mathbb{B}_{\mathbb{V}}(f_i, f_j) = 0, \text{ and } \mathbb{B}_{\mathbb{V}}(e_i, f_j) = \delta_{ij} \ \forall\ 1 \leq i, j \leq n,$$

where δ_{ij} is the Kronecker symbol. Under the isomorphism $\mathbb{V} \cong \mathrm{F}^{2n}$ under the above basis, we have

$$\mathbb{B}_{\mathbb{V}} : \mathrm{F}^{2n} \times \mathrm{F}^{2n} \to \mathrm{F} : \left(\begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix} \right) \mapsto \sum_{i=1}^{2n} x_i y_{2n+1-i}.$$

Take $\mathbb{W}$ to be a 2-dimensional symplectic space over F, and fix the ordered basis $\{e, f\}$ of $\mathbb{W}$ such that $\mathbb{B}_{\mathbb{W}}(e, f) = 1$. Under the isomorphism $\mathbb{W} \cong \mathrm{F}^2$ under this fixed basis, we have

$$\mathbb{B}_{\mathbb{W}} : \mathrm{F}^2 \times \mathrm{F}^2 \to \mathrm{F} : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto x_1 y_2 - x_2 y_1.$$

In this case, $\mathrm{U}(\mathbb{V}, \mathbb{B}_{\mathbb{V}}) = \mathrm{SO}(\mathbb{V})$ and $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{SL}_2$. Recall that we take

$$v_0 = e_1 + f_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathrm{F}^{2n}$$

and

$$\mathrm{X} = \mathrm{SO}(\mathbb{V})/\mathrm{SO}(v_0^{\perp}) \cong \{v \in \mathbb{V} \mid \mathbb{B}_{\mathbb{V}}(v,v) = 2\} \subset \mathbb{V} : g \mapsto gv_0.$$

Let Φ_0 be the characteristic function of $\mathbb{Y}(\mathfrak{o}_{\mathrm{F}})$.

2.4.1 Matching of unit elements

Lemma 2.4.1.

$$\mathcal{I}(\mathrm{Res}(\Phi_0))(x)\,\mathrm{d}x = \frac{1}{\zeta_{\mathrm{F}}(n)}\mathbb{L}_{\mathrm{X}}^{\circ} = \frac{\#\mathrm{X}(\kappa_{\mathrm{F}})}{q^{\dim \mathrm{X}}}\mathbb{L}_{\mathrm{X}}^{\circ}.$$

Proof. Denote

$$F(x) := \int_{\mathbb{V}_x} \Phi_0(v)\omega_x(v),\ x \neq 0.$$

Then we only need to compute $F(1)$. To do this, let

$$F^*(x) := \int_{\mathbb{V}} \Phi_0(v)\psi\left(x \cdot \frac{\tau(\mathbb{B}_{\mathbb{V}}(v,v))}{2}\right)\mathrm{d}v.$$

According to [Igu00, Section 8.3, Theorem 8.3.1], we know that

$$F^*(x) = \max\{1, |x|\}^{-n},$$

and

$$\begin{aligned} F(0) &= \lim_{k\to\infty}\int_{|x|\leq q^k} F^*(x)\psi(-x)\,\mathrm{d}x \\ &= \int_{|x|\leq 1} 1\,\mathrm{d}x + \lim_{k\to\infty}\int_{q\leq |x|\leq q^k} |x|^{-n}\psi(-x)\,\mathrm{d}x \\ &= 1 - q^{-n}. \end{aligned}$$

On the other hand, due to [Tay92, Chapter 11],

$$\#\mathrm{O}(\mathbb{V})(\kappa_{\mathrm{F}}) = 2q^{n(n-1)}(q^n - 1)\prod_{i=0}^{n-1}(q^{2i} - 1)$$

and

$$\#\mathrm{O}(v_0^{\perp})(\kappa_{\mathrm{F}}) = 2q^{(n-1)^2}\prod_{i=0}^{n-1}(q^{2i} - 1),$$

hence

$$\frac{\#\mathrm{X}(\kappa_{\mathrm{F}})}{q^{\dim \mathrm{X}}} = 1 - q^{-n} = \zeta_{\mathrm{F}}(n)^{-1}.$$

∎

Lemma 2.4.2.
$$\Omega(\Phi_0) = \frac{1}{\zeta_{\mathrm{F}}(2n-2)}\Phi_{L_{\mathrm{X}}}.$$

Proof. The identity follows from the fact that both functions are (N, ψ)-left invariant and $\mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})$-right invariant; since they agree on the torus T, they must coincide everywhere by the Iwasawa decomposition. □

As a consequence,

Theorem 2.4.3 (fundamental lemma for the unit element)**.** *Let*

$$\mathbb{L}_{\mathrm{X}}^{\circ} \in \mathcal{S}(\mathrm{SO}_{2n-1}\backslash \mathrm{SO}_{2n}/\mathrm{SO}_{2n-1})$$

be the basic measure on $\mathbb{A}^1$*, then we have*

$$\mathcal{T}^{-1}(\mathbb{L}_{\mathrm{X}}^{\circ}) = \frac{q^{\dim \mathrm{X}}}{\#\overline{\mathrm{X}}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})} f_{L_{\mathrm{X}}} = \frac{\zeta_{\mathrm{F}}(n)}{\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(2n-2)} f_{L(\mathrm{Ad}, n-1)}$$

2.4.2 Matching of Hecke elements

Let B be the Borel subgroup of $\mathrm{SO}(\mathbb{V})$ fixing the isotropic flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \cdots, e_n\rangle,$$

and A the maximal torus consisting of elements fixing each line $\langle e_i\rangle$ for every $1 \le i \le n$.

Lemma 2.4.4.
$$\mathrm{X}^{\circ} = \{v \in \mathrm{X} \mid \mathbb{B}_{\mathbb{V}}(x, e_1) \neq 0\}.$$

Then $v_0 \in \mathrm{X}^{\circ}$ *and* X° *is an open* B*-orbit in* X*.*

Proof. It is clear that X° is open in X, stable under the action of B and $v_0 \in \mathrm{X}^{\circ}$. So we only need to show $\mathrm{X}^{\circ} = \mathrm{B}v_0$.

Indeed, let $x = \sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n} y_i f_i \in \mathrm{X}^{\circ}$, then $y_1 \neq 0$. By the action of A, we may assume $y_1 = 1$. Under the ordered basis $\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$, we have

$$\begin{pmatrix} s_n \left({}^t\begin{pmatrix} 1 & & & -y_n \\ & 1 & & -y_{n-1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \right)^{-1} s_n & 0_n \\ 0_n & \begin{pmatrix} 1 & & & -y_n \\ & 1 & & -y_{n-1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y_n \\ y_{n-1} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_2' \\ \vdots \\ x_n' \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

for some $x_2', \cdots, x_n' \in \mathrm{F}$. And here $s_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. Then one can take

$$\begin{pmatrix} \mathrm{I}_n & \begin{pmatrix} x_n' & x_{n-1}' & \cdots & x_2' & 0 \\ 0 & 0 & \cdots & 0 & -x_2' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -x_{n-1}' \\ 0 & 0 & 0 & 0 & -x_n' \end{pmatrix} \\ 0_n & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} 1 \\ x_2' \\ \vdots \\ x_n' \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

$\square$

Corollary 2.4.5. $\mathrm{P}(\mathrm{X})$ *is the maximal proper parabolic subgroup fixing the isotropic flag*

$$\langle e_1 \rangle.$$

Then under the above basis $\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$, *the Levi* $\mathrm{L}(\mathrm{X})$ *of* $\mathrm{P}(\mathrm{X})$ *is*

$$\mathrm{L}(\mathrm{X}) = \left\{ \begin{pmatrix} t & & \\ & g & \\ & & t^{-1} \end{pmatrix} \mid t \in \mathbb{G}_m, g \in \mathrm{SO}(v_0^{\perp}) \right\} \cong \mathbb{G}_m \times \mathrm{SO}(v_0^{\perp}),$$

and hence

$$\delta_{(\mathrm{X})}^{\frac{1}{2}}\left(\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}\right) = |t_2|^{n-2}|t_3|^{n-3}\cdots|t_n|^0.$$

Moreover, the quotient map $\mathrm{A} \to \mathrm{A}_{\mathrm{X}} \cong \mathbb{G}_m$ *is given by*

$$\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto t_1.$$

Then the normalized map $\mathbb{C}[\mathrm{A}^\vee]^{\mathbb{W}_{\mathrm{G}}} \to \mathbb{C}[\mathrm{A}_{\mathrm{X}}^\vee]^{\mathbb{W}_{\mathrm{X}}}$ *is given by*

$$\mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathbb{W}_{\mathrm{SO}_{2n}}} \to \mathbb{C}[x^{\pm}]^{\mathbb{Z}/2\mathbb{Z}} : x_1 \mapsto x,\ x_j \mapsto q^{j-n} \text{ for } 2 \leq j \leq n. \tag{2.8}$$

Now consider the nilpotent cone $\Sigma := \{v \in \mathbb{V} \mid \mathbb{B}_{\mathbb{V}}(v,v) = 0\}$.

Lemma 2.4.6 (Lemma 3.1 in [Ral82])**.**

$$\Sigma^\circ := \left\{ g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid g \in \mathrm{SO}(\mathbb{V}) \right\}$$

is an open dense $\mathrm{SO}(\mathbb{V})$*-orbit of* Σ*.*

Proof. According to Witt's Theorem, $\Sigma = \Sigma^\circ \bigsqcup \{0\}$, see [Ser73, Chapter 4]. See also [Ral82, Lemma 3.1]. $\square$

Let ω_ψ be the Weil representation of $\mathrm{SO}(\mathbb{V}) \times \mathrm{SL}_2$ on $\mathcal{F}(\mathbb{Y})$.

Lemma 2.4.7 (Proposition 2.2 in [Ral82])**.** *Let* $\Phi \in \mathcal{F}(\mathbb{Y})^{\mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})}$*, and* $\omega_\psi(g)\Phi|_\Sigma = 0$ *for all* $g \in \mathrm{SL}_2$*, then* $\Phi = 0$*.*

Proof. This is a special case of [Ral82, Proposition 2.2]. But let us include the proof here for the convenience of the reader. Note that as a representation of SL_2, the dual of the Jacquet module

$$\mathcal{F}(\mathbb{Y})_{\mathrm{N}} := \mathcal{F}(\mathbb{Y}) \Big/ \left\langle \Phi - \omega_\psi(n)\Phi \mid \Phi \in \mathcal{F}(\mathbb{Y}), n \in \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \right\rangle$$

is isomorphic to the distributions on $\mathbb{Y}$ supported on Σ. Then the space of functions vanishing on Σ is isomorphic to the Jacquet module $\mathcal{F}(\mathbb{Y})_{\mathrm{N}}$. Since for any $g \in \mathrm{SL}_2$, $\omega_\psi(g)\Phi|_\Sigma = 0$, the SL_2-module generated by Φ is cuspidal. But we know any such module cannot be unramified, hence $\Phi = 0$. $\square$

Now let $\sigma \in \mathbb{C}$ be such that $\mathrm{Re}(\sigma) > 1$. Consider the following intertwining operators introduced in [Ral82]:

$$\mathrm{Z}_\sigma : \mathcal{F}(\mathbb{Y}) \to \mathcal{C}^\infty(\mathrm{SO}_{2n} \times \mathrm{SL}_2) : \Phi \mapsto (g_1, g_2) \mapsto \int_{\mathrm{F}} \omega_\psi(g_1, g_2)^{-1} \Phi \left(\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) |a|^\sigma \frac{\mathrm{d}a}{|a|},$$

which is absolutely convergent according to [Tat67].

Proposition 2.4.8 (Lemma 4.1 and Remark 4.4 in [Ral82])**.** *When* $\mathrm{Re}(\sigma) > 1$*,*

(1) Let $\mathfrak{s} = (s_1, s_2, \cdots, s_n) \in X^*(\mathrm{A}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^n$*, write*

$$\chi(\mathfrak{s}) : \mathrm{T} \to \mathbb{C} : \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto \prod_{i=1}^n |t_i|^{s_i}$$

and $\mathrm{I}(\chi(\mathfrak{s}))$ *the correponding normalized induced representation of* SO_{2n}*. Then* Z_σ *factors through* $\mathrm{I}(\chi(n-1-\sigma, n-2, n-3, \cdots, 0)) \otimes \mathrm{I}(\chi(\sigma - n + 1))$ *as representations of* $\mathrm{SO}_{2n} \times \mathrm{SL}_2$*.*

(2) *Write* K *for the maximal compact subgroup of* $\mathrm{SO}(\mathbb{V})$ *fixing the standard lattice* $\mathfrak{o}_{\mathrm{F}}^{2n}$ *under the ordered basis* $\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$*. Let*

$$\lambda_{\mathrm{X}} : \mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K}) \to \mathcal{H}(\mathrm{SL}_2, \mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}}))$$

be the morphism of the Hecke algebra corresponding to (2.8). Let Φ_0 *be the characteristic function of* $\mathbb{Y}(\mathfrak{o}_{\mathrm{F}})$*, then we have*

$$\omega_\psi(h^\vee)\Phi_0 = \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0.$$

for all $h \in \mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K})$*.*

Proof. (1) is simply a matter of computing the effect of $\mathrm{A} \times \mathrm{T}$ on Z_σ when $\mathrm{Re}(\sigma) > 1$.

As for (2), a first observation is that for any $f \in \mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K})$,

$$\begin{aligned} &\mathrm{Z}_\sigma(\omega_\psi(h^\vee)\Phi_0 - \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0) \\ &= \big(\mathrm{Sat}(h^\vee)(n-1-\sigma, n-2, \cdots, 0) - \mathrm{Sat}(\lambda_{\mathrm{X}}(h))(\sigma - n + 1)\big)\, \mathrm{Z}_\sigma(\Phi_0) = 0 \end{aligned}$$

when $\mathrm{Re}(\sigma) > 1$. Hence for any $g_1 \in \mathrm{SO}(\mathbb{V})$ and $g_2 \in \mathrm{SL}_2$, as a function on F,

$$a \mapsto \omega_\psi(g_1, g_2)^{-1}(\omega_\psi(h^\vee)\Phi_0 - \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0)\left(\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$$

is $\mathfrak{o}_{\mathrm{F}}^\times$-invariant and its Mellin transform is zero. Hence according to the Mellin inversion [Igu78, Chapter 1],

$$\omega_\psi(g_1, g_2)^{-1}(\omega_\psi(h^\vee)\Phi_0 - \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0)\left(\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$$

for all $a \in \mathrm{F}$. By varing $g_1 \in \mathrm{SO}(\mathbb{V})$ and Lemma 2.4.6, we see for any $g_2 \in \mathrm{SL}_2$,

$$\omega_\psi(g_2)(\omega_\psi(h^\vee)\Phi_0 - \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0)|_\Sigma = 0.$$

Now as $\omega_\psi(h^\vee)\Phi_0 - \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0 \in \mathcal{F}(\mathbb{V})^{\mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})}$, according to Lemma 2.4.7, we obtain that

$$\omega_\psi(h^\vee)\Phi_0 = \omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0.$$

∎

Theorem 2.4.9. *Conjecture 2.1.5 is true for* X *of type* D_n.

Proof. As the restriction map $\mathcal{F}(\mathbb{V}) \to \mathcal{F}(\mathrm{X})$ is obviously $\mathrm{SO}(\mathbb{V})$-equivariant, hence for any $h \in \mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K})$,

$$\mathcal{I} \circ \mathrm{res}(\omega_\psi(h^\vee)\Phi_0)\,\mathrm{d}x = \mathcal{I}(h \star (\mathrm{res}(\Phi_0)))\,\mathrm{d}x = h \star \frac{1}{\zeta_{\mathrm{F}}(n)}\mathbb{L}_{\mathrm{X}}^{\circ}.$$

On the other hand, as functions on SL_2, since we know from Lemma 2.4.2

$$\Omega(\Phi_0)(g) = \omega_\psi(g)\Phi_0(v_0) = \frac{1}{\zeta_{\mathrm{F}}(2n-2)}\Phi_{L(\mathrm{Ad},n-1)}(g),$$

hence we have

$$\begin{aligned}\Omega(\omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0) &= (\lambda_{\mathrm{X}}(h))^\vee \star \frac{1}{\zeta_{\mathrm{F}}(2n-2)}\Phi_{L(\mathrm{Ad},n-1)}(g) \\ &= \lambda_{\mathrm{X}}(h) \star \frac{1}{\zeta_{\mathrm{F}}(2n-2)}\Phi_{L(\mathrm{Ad},n-1)}(g).\end{aligned}$$

Therefore, Conjecture 2.1.5 is true for X of type D_n. $\square$

2.5 Fundamental lemma of type A_{n-1}

In this section, take $\mathbb{D} = \mathrm{F} \times \mathrm{F}$, $\mathbb{V} = \mathrm{V} \times \mathrm{V}^\vee$, where V is a vector space over F and $\mathrm{V}^\vee$ its dual space. Fix an ordered basis $\{e_1, \cdots, e_n\}$ of F^n and its dual basis $\{e_1^\vee, \cdots, e_n^\vee\}$ of $\mathrm{V}^\vee$, then we have an isomorphism $\mathbb{V} \cong \mathrm{F}^n \times \mathrm{F}^n$, and take

$$\mathbb{B}_{\mathbb{V}} : \mathrm{F}^n \times \mathrm{F}^n \to \mathrm{F} \times \mathrm{F}$$
$$\left(\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right), \left(\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix}\right)\right) \mapsto \left(\sum_{i=1}^n x_i\zeta_i, \sum_{i=1}^n y_i\xi_i\right).$$

Similarly take $\mathbb{W} = \mathrm{W} \times \mathrm{W}^\vee$, where W is a 2-dimensional vector space over F. Fix an ordered basis $\{f_1, f_2\}$ of W and its dual basis $\{f_1^\vee, f_2^\vee\}$ of $\mathrm{W}^\vee$, then we have an isomorphism $\mathbb{W} \cong \mathrm{F}^2 \times \mathrm{F}^2$, and take

$$\mathbb{W} : \mathrm{F}^2 \times \mathrm{F}^2 \to \mathrm{F} \times \mathrm{F}$$
$$\left(\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right), \left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\right)\right) \mapsto \left(\sum_{i=1}^2 x_i\zeta_i, -\sum_{i=1}^2 y_i\xi_i\right).$$

Write $e := (f_1, f_2^\vee)$ and $f := (-f_2, f_1^\vee)$. In this case, $\mathrm{U}(\mathbb{V}, \mathbb{B}_\mathbb{V}) = \mathrm{GL}(\mathrm{V}) = \mathrm{GL}_n$ and $\mathrm{U}(\mathbb{W}, \mathbb{B}_\mathbb{W}) = \mathrm{GL}(\mathrm{W}) = \mathrm{GL}_2$. Recall that $v_0 = e_n + e_n^\vee$, and we have

$$\mathrm{X} = \mathrm{GL}_n/\mathrm{GL}_{n-1} \cong \{v \in \mathbb{V} \mid \mathrm{tr}\, \mathbb{B}_\mathbb{V}(v, v) = 2\} \subset \mathbb{V} : g \mapsto gv_0,$$

where we identify $\mathrm{U}(v_0^\perp) = \mathrm{GL}_{n-1}$. Let $\overline{\mathrm{X}} := \mathrm{X}/\mathbb{G}_m = \mathrm{PGL}_n/\mathrm{GL}_{n-1}$, which classifies the decompositions $\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_{n-1}$, the direct sum of V as 1-dimensional and $n-1$-dimensional subspaces. Let Φ_0 be the characteristic function of $\mathbb{Y}(\mathfrak{o}_\mathrm{F})$.

2.5.1 Matching of unit elements

Lemma 2.5.1. *The push-forward of $\mathcal{I}(\mathrm{Res}(\Phi_0))(x)\,\mathrm{d}x$ in $\mathbb{A}^1$ is*

$$\frac{1}{\zeta_\mathrm{F}(n)}\mathbb{L}^\circ_{\mathrm{X}} = \frac{q^{\dim \overline{\mathrm{X}}}}{\#\overline{\mathrm{X}}(\kappa_\mathrm{F})}\mathbb{L}^\circ_{\mathrm{X}}.$$

Proof. The first part follows from the same argument as in Lemma 2.4.1. It is well known that

$$\frac{q^{\dim \overline{\mathrm{X}}}}{\#\overline{\mathrm{X}}(\kappa_\mathrm{F})} = \frac{1}{\zeta_\mathrm{F}(n)}.$$

□

Lemma 2.5.2.

$$\int_{\mathbb{G}_m} \Omega(\Phi_0)(gz)\,\mathrm{d}z = \frac{1}{\zeta_\mathrm{F}(1)\zeta_\mathrm{F}(n-1)}\Phi_{L(\mathrm{Std}, \frac{\mathrm{n}-1}{2})(\mathrm{Std}, \frac{\mathrm{n}-1}{2})}(g).$$

Proof. We only need to check that both sides match on the maximal torus. Note that

$$\begin{aligned}
\int_{\mathbb{G}_m} \Omega(\Phi_0)\left(\begin{pmatrix} \xi & \\ & 1\end{pmatrix}\begin{pmatrix} z & \\ & 1\end{pmatrix}\right)\mathrm{d}z &= \int_{\mathbb{G}_m} |\xi|^{\frac{n}{2}} 1_{\mathfrak{o}_\mathrm{F}}(\zeta z) 1_{\mathfrak{o}_\mathrm{F}}(z^{-1})\,\mathrm{d}z \\
&= (1-q^{-1})|\xi|^{\frac{n}{2}}(1 - \log_q|\xi|) \\
&= \frac{1}{\zeta_\mathrm{F}(1)\zeta_\mathrm{F}(n-1)}\Phi_{L(\mathrm{Std}, \frac{\mathrm{n}-1}{2})(\mathrm{Std}, \frac{\mathrm{n}-1}{2})}\left(\begin{pmatrix} \xi & \\ & 1\end{pmatrix}\right).
\end{aligned}$$

□

Therefore,

Theorem 2.5.3. *Let* $\mathbb{L}_{\mathrm{X}}^{\circ} \in \mathcal{S}(\mathrm{GL}_{n-1}\backslash\mathrm{PGL}_n/\mathrm{GL}_{n-1})$ *be the push-forward of the probability measure which is equal to the characteristic function of* $\overline{\mathrm{X}}(\mathfrak{o}_{\mathrm{F}}) \times \overline{\mathrm{X}}(\mathfrak{o}_{\mathrm{F}})$ *times an invariant measure, then we have*

$$\mathcal{T}^{-1}\left(\mathbb{L}_{\mathrm{X}}^{\circ}\right) = \frac{q^{\dim \mathrm{X}}}{\#\overline{\mathrm{X}}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})} f_{L_{\mathrm{X}}} = \frac{\zeta_{\mathrm{F}}(n)}{\zeta_{\mathrm{F}}(1)\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(n-1)} f_{L\left(\mathrm{Std},\frac{n-1}{2}\right)\left(\mathrm{Std},\frac{n-1}{2}\right)}.$$

2.5.2 Matching of Hecke elements

Let B be the Borel subgroup of GL_n consisting of all upper triangular matrices and A the maximal torus consisting of diagonal matrices under the ordered basis $\{e_1, \cdots, e_n\}$. Let $\overline{\mathrm{B}}$ and $\overline{\mathrm{A}}$ be their images in PGL_n.

Lemma 2.5.4. *Let*

$$\overline{\mathrm{X}}^{\circ} := \{\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_{n-1} \mid \mathrm{V}_1 \not\subseteq \langle e_1, \cdots, e_{n-1}\rangle, e_1 \notin \mathrm{V}_{n-1}\}.$$

Then $\overline{\mathrm{X}}^{\circ}$ *is an open* $\overline{\mathrm{B}}$*-orbit in* $\overline{\mathrm{X}}$.

Proof. Note that if we identify V_{n-1} with its normal direction $\mathrm{F}\mathbf{n}$ with $\mathbf{n} \in \mathrm{V}^{\vee}$, then the action of $\mathrm{GL}(\mathrm{V})$ on $\mathbf{n}$ is given by the transpose. Hence, the Lemma follows easily. At this time the decomposition $\mathrm{V} = \mathrm{F}e_n \oplus \{v \in \mathrm{V} \mid \langle v, e_1^{\vee}\rangle = 0\}$ lies in $\overline{\mathrm{X}}^{\circ}$. □

Corollary 2.5.5. *Let* $\mathrm{P}(\overline{\mathrm{X}})$ *be the parabolic subgroup of* PGL_n *stabilizing* $\overline{\mathrm{X}}^{\circ}$*, then its Levi*

$$\mathrm{L}(\overline{\mathrm{X}}) = \left\{ \begin{pmatrix} t_1 & & \\ & g & \\ & & t_2 \end{pmatrix} \mid t_1, t_2 \in \mathbb{G}_m, g \in \mathrm{GL}_{n-2} \right\} / \mathbb{G}_m.$$

Therefore,

$$\delta_{(\overline{\mathrm{X}})}^{\frac{1}{2}} \left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \right) = |a_2|^{\frac{n-3}{2}} |a_3|^{\frac{n-5}{2}} \cdots |a_{n-1}|^{\frac{3-n}{2}}.$$

Then $\mathbb{C}[\overline{\mathrm{A}}^{\vee}]^{\mathbb{W}_{\mathrm{PGL}_n}} \to \mathbb{C}[\mathrm{A}_{\overline{\mathrm{X}}}^{\vee}]^{\mathbb{W}_{\overline{\mathrm{X}}}}$ *is given by*

$$\begin{gathered} \left(\mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]/(x_1 \cdots x_n - 1)\right)^{\mathfrak{S}_n} \to \mathbb{C}[x^{\pm}]^{\mathbb{Z}/2\mathbb{Z}} \\ x_1 \mapsto x, x_n \mapsto x^{-1}, x_j \mapsto q^{\frac{2j-n-1}{2}} \ (2 \leq j \leq n-1), \end{gathered} \tag{2.9}$$

where $\mathfrak{S}_n$ *is the permutation group of of* $(1, 2, 3, \cdots, n)$.

Consider the nilpotent cone $\Sigma := \{(v, v^\vee) \in \mathbb{V} \mid \langle v, v^\vee \rangle = 0\}$ of $\mathbb{V}$.

Lemma 2.5.6.
$$\Sigma^\circ := \left\{ \left(g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, {}^t g^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right) \mid g \in \mathrm{GL}_n \right\}$$
is an open GL_n*-orbit of* Σ.

Proof. It is clear that $\dim \Sigma = 2n-1$. On the other hand, since the stabilizer of $(e_1, e_n^\vee)$ is
$$\left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \right\},$$
hence $\dim \Sigma^\circ = n^2 - (n^2 - (2n-1)) = 2n - 1 = \dim \Sigma$. $\square$

Let ω_ψ be the Weil representation of $\mathrm{GL}_n \times \mathrm{GL}_2$ on $\mathcal{F}(\mathbb{Y})$.

Lemma 2.5.7. *Let* $\Phi \in \mathcal{F}(\mathbb{Y})^{\mathrm{GL}_2(\mathfrak{o}_\mathrm{F})}$*, and* $\omega_\psi(g)\Phi|_\Sigma = 0$ *for all* $g \in \mathrm{GL}_2$*, then* $\Phi = 0$.

Proof. Similarly to 2.4.7, it suffices to show that the dual of the Jacquet module
$$\mathcal{F}(\mathbb{Y})_\mathrm{N} := \mathcal{F}(\mathbb{Y}) / \left\langle \Phi - \omega_\psi(n)\Phi \mid \Phi \in \mathcal{F}(\mathbb{Y}), n \in \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \right\rangle$$
is isomorphic to the distributions on $\mathbb{Y}$ supported on Σ. Let φ be a distribution on $\mathcal{F}(\mathbb{Y})$ that vanishes on every $\Phi - \omega_\psi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right)\Phi$ for $\Phi \in \mathcal{F}(\mathbb{Y})$ and $n \in \mathrm{F}$. Since for $y \in \mathbb{Y}$, we have
$$\omega_\psi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right)\Phi(y) = \psi\left(\frac{n \cdot \tau(\mathbb{B}(y,y))}{2}\right)\Phi(y),$$
then
$$\varphi(y) - \psi\left(\frac{n \cdot \tau(\mathbb{B}(y,y))}{2}\right)\varphi(y) = 0$$
as distributions for any $n \in \mathrm{F}$. Then φ is supported $\{y \in \mathbb{Y} \mid \tau(\mathbb{B}(y,y)) = 0\} = \Sigma$. $\square$

Now let $\sigma_1, \sigma_2 \in \mathbb{C}$ be such that $\mathrm{Re}(\sigma_1), \mathrm{Re}(\sigma_2) > 1$, and consider the following intertwining operators

$$\mathrm{Z}_{\sigma_1,\sigma_2} : \mathcal{F}(\mathbb{Y}) \to \mathcal{C}^\infty(\mathrm{GL}_n \times \mathrm{GL}_2) : \Phi \mapsto \int_{\mathrm{F}\times\mathrm{F}} \omega_\psi(g_1, g_2)^{-1} \Phi\left(\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix} \right) |a|^{\sigma_1} |b|^{\sigma_2} \frac{\mathrm{d}a}{|a|} \frac{\mathrm{d}b}{|b|},$$

which is absolutely convergent.

Proposition 2.5.8. *When* $\mathrm{Re}(\sigma_1), \mathrm{Re}(\sigma_2) > 1$*,*

(1) Let $\mathfrak{s} = (s_1, s_2, \cdots, s_n) \in X^*(\mathrm{A}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$*, write*

$$\chi(\mathfrak{s}) : \mathrm{A} \to \mathbb{C} : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^n |t_i|^{s_i},$$

and $\mathrm{I}(\chi(\mathfrak{s}))$ *the correponding normalized induced representation of* GL_n*. Then* $\mathrm{Z}_{\sigma_1,\sigma_2}$ *factors through* $\mathrm{I}(\chi(\frac{n-1}{2} - \sigma_1, \frac{n-3}{2}, \cdots, \frac{3-n}{2}, \frac{1-n}{2} + \sigma_2)) \otimes \mathrm{I}(\chi(\sigma_1 - \frac{n-1}{2}, -\sigma_2 + \frac{n-1}{2}))$*.*

(2) Let $\lambda_{\mathrm{X}} : \mathcal{H}(\mathrm{GL}_n, \mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}})) \to \mathcal{H}(\mathrm{GL}_2, \mathrm{GL}_2(\mathfrak{o}_{\mathrm{F}}))$ *be the morphism between Hecke algebra whose Stakee transform is given by*

$$\mathbb{C}[x_1^\pm, \cdots, x_n^\pm]^{\mathfrak{S}_n} \to \mathbb{C}[x_1^\pm, x_2^\pm]^{\mathfrak{S}_2} : x_1 \mapsto x_1, x_n \mapsto x_2, x_j \mapsto q^{\frac{2j-n-1}{2}} \ (2 \leq j \leq n-1).$$

Then we have $\omega_\psi(f^\vee)\Phi_0 = \omega_\psi(\lambda_{\mathrm{X}}(f))\Phi_0$*.*

Proof. It follows from the same argument as in Proposition 2.4.8. $\square$

Theorem 2.5.9. *Conjecture 2.1.5 is true for* X *of type* A_{n-1}*.*

Proof. Consider the natural homomorphism $\mathbb{C}[\mathrm{A}^\vee]^{\mathbb{W}_{\mathrm{GL}_n}} \to \mathbb{C}[\overline{\mathrm{A}}^\vee]^{\mathbb{W}_{\mathrm{PGL}_n}}$, and the corresponding homomorphism between Hecke algebras

$$\mathcal{H}(\mathrm{GL}_n, \mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}})) \to \mathcal{H}(\mathrm{PGL}_n, \mathrm{PGL}_n(\mathfrak{o}_{\mathrm{F}})).$$

Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathrm{GL}_n, \mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}})) \arrow[r, "\lambda_{\mathrm{X}}"] \arrow[d, "\alpha"] & \mathcal{H}(\mathrm{GL}_2, \mathrm{GL}_2(\mathfrak{o}_{\mathrm{F}})) \arrow[d, "\beta"] \\
\mathcal{H}(\mathrm{PGL}_n, \mathrm{PGL}_n(\mathfrak{o}_{\mathrm{F}})) \arrow[r, "\lambda_{\overline{\mathrm{X}}}"] & \mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})),
\end{tikzcd}$$

where the bottom arrow is given by (2.9). According to Lemma 2.3.11, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{GL}_n/\mathrm{GL}_{n-1} \arrow[r] \arrow[d, "p"] & \mathbb{A}^2 \arrow[d, "q"] \\
\mathrm{PGL}_n/\mathrm{GL}_{n-1} \arrow[r, "\pi"] & \mathbb{A}^1
\end{tikzcd},$$

where q is given by $(x, y) \mapsto xy$, and π can be identified with the canonical quotient map

$$\mathrm{PGL}_n/\mathrm{GL}_{n-1} \to \mathrm{GL}_{n-1} \backslash\!\backslash \mathrm{PGL}_n/\mathrm{GL}_{n-1}.$$

For simplicity of notation, let us identify functions with measures using the Haar measures we specified, and let $(\cdot)_*$ denote the push-forward of measures. Let $h \in \mathcal{H}(\mathrm{GL}_n, \mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}}))$. Then we have

$$\mathcal{T}^{-1}(\alpha(h) \star \mathbb{L}_{\mathrm{X}}^{\circ}) = \mathcal{T}^{-1}\left(\zeta_{\mathrm{F}}(n) \cdot \pi_* \circ p_*(h \star \mathrm{res}(\Phi_0))\right).$$

According to [Sat63, Section 7],

$$\pi_* \circ p_*(h \star \mathrm{res}(\Phi_0)) = \pi_* \circ p_*(\omega_\psi(h^\vee)(\mathrm{res}(\Phi_0))),$$

which is equal to

$$q_* \circ \mathcal{I}(\omega_\psi(h^\vee)(\mathrm{res}(\Phi_0))) = q_* \circ \mathcal{I}(\omega_\psi(\lambda_{\mathrm{X}}(h))\mathrm{res}(\Phi_0))$$

according to Proposition 2.5.8.

Due to Lemma 2.5.2, we have

$$\int_{\mathrm{Z}} \omega_\psi(gz)\Phi_0(v_0)\,\mathrm{d}z = \frac{1}{\zeta_{\mathrm{F}}(1)\zeta_{\mathrm{F}}(n-1)} \Phi_{L(\mathrm{Std}, \frac{\mathrm{n}-1}{2})(\mathrm{Std}, \frac{\mathrm{n}-1}{2})}(g), \ \forall g \in \mathrm{PGL}_2,$$

then for $g \in \mathrm{PGL}_2$,

$$\begin{aligned}
\int_{\mathrm{Z}} \omega_\psi(gz)\omega_\psi(\lambda_{\mathrm{X}}(h))\Phi_0(v_0)\,\mathrm{d}z &= \left(\int_{\mathrm{Z}} \lambda_{\mathrm{X}}(h) \star \omega_\psi(\cdot z)\Phi_0(v_0)\,\mathrm{d}z\right)(g) \\
&= \beta(\lambda_{\mathrm{X}}(h)) \star \frac{1}{\zeta_{\mathrm{F}}(1)\zeta_{\mathrm{F}}(n-1)} \Phi_{L(\mathrm{Std}, \frac{\mathrm{n}-1}{2})(\mathrm{Std}, \frac{\mathrm{n}-1}{2})}(g)
\end{aligned}$$

due to [Sat63, Section 7]. Finally because α is surjective and $\beta(\lambda_{\mathrm{X}}(h)) = \lambda_{\overline{\mathrm{X}}}(\alpha(h))$, we obtain that

$$\mathcal{T}^{-1}(h \star \mathbb{L}_{\mathrm{X}}^{\circ}) = \lambda_{\mathrm{X}}(h) \star \frac{\zeta_{\mathrm{F}}(n)}{\zeta_{\mathrm{F}}(1)\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(n-1)} f_{L(\mathrm{Std}, \frac{\mathrm{n}-1}{2})(\mathrm{Std}, \frac{\mathrm{n}-1}{2})}.$$

$\square$

2.6 Fundamental lemma of type C_n

In this section, let $\mathbb{D} = \mathrm{Mat}_2(\mathrm{F})$, and fix the isomorphism $\mathrm{F}^2 \otimes_{\mathrm{F}} \mathrm{F}^2 \cong \mathbb{D}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes (c, d) \mapsto \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}.$$

Take $\mathbb{V} = \mathbb{D}^n$, and

$$\mathbb{B}_{\mathbb{V}} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{D} : ((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \mapsto \sum_{i=1}^{n} \overline{x_i} y_i.$$

Under the above identification $\mathbb{D} \cong \mathrm{F}^2 \otimes_{\mathrm{F}} \mathrm{F}^2$, and $\mathbb{D}^n \cong \mathrm{F}^{2n} \otimes_{\mathrm{F}} \mathrm{F}^2$, the above form is given by

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_{2n} \end{pmatrix} \otimes (x_1, x_2), \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} \otimes (y_1, y_2) \right)$$

$$\mapsto (v_1, \cdots, v_{2n}) \cdot \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} \cdot J^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (y_1, y_2),$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly take $\mathbb{W} = \mathbb{D}^2$ and

$$\mathbb{B}_{\mathbb{W}} : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{D} : ((x_1, x_2), (y_1, y_2)) \mapsto \overline{x_1} y_2 - \overline{x_2} y_1.$$

Then under the identification $\mathbb{D}^2 \cong \mathrm{F}^4 \otimes_{\mathrm{F}} \mathrm{F}^2$ as above, this form is given by

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_4 \end{pmatrix} \otimes (x_1, x_2), \begin{pmatrix} w_1 \\ \vdots \\ w_4 \end{pmatrix} \otimes (y_1, y_2)\right)$$

$$\mapsto (v_1, \cdots, v_4) \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_4 \end{pmatrix} \cdot J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (y_1, y_2).$$

Let $e := (\mathrm{I}_2, 0)$ and $f := (0, \mathrm{I}_2)$. In this case, we have $\mathrm{U}(\mathbb{V}, \mathbb{B}_{\mathbb{V}}) = \mathrm{Sp}_{2n}$ and $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{O}_4$. To simplify the notation, let $\{e_1, f_1, \cdots, e_n, f_n\}$ denote the standard basis of F^{2n}. Specifically, for $1 \leq i \leq n$, let e_i and f_i be the column vectors with a 1 in the $(2i-1)$-th and $2i$-th positions, respectively, and zero elsewhere.

We have $\mathrm{G} = \mathrm{U}(\mathbb{V}, \mathbb{B}_{\mathbb{V}}) = \mathrm{Sp}_{2n}$, and $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{O}_4$, and

$$\mathrm{X} = \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \cong \{v \in \mathbb{V} \mid \mathrm{tr}\, \mathbb{B}_{\mathbb{V}}(v, v) = 2\} \subset \mathbb{V} : g \mapsto g v_0,$$

where $v_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I}_2 \end{pmatrix} \in \mathbb{D}^n$. Let $\overline{\mathrm{X}} = \mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2$, which classifies the 2-dimensional non-degerate subspaces V_2 of F^{2n}. Let Φ_0 be the characteristic function of $\mathbb{Y}(\mathfrak{o}_{\mathrm{F}})$,

2.6.1 Matching of unit elements

Lemma 2.6.1. *The push-forward of* $\mathcal{I}(\mathrm{Res}(\Phi_0))(x)\,\mathrm{d}x$ *in* $\mathcal{S}(\overline{\mathfrak{X}})$ *is*

$$\frac{1}{\zeta_{\mathrm{F}}(2n)} \mathbb{L}^{\circ}_{\mathrm{X}} = \frac{\#\mathrm{X}(\kappa_{\mathrm{F}})}{q^{\dim_{\mathrm{X}}}} \mathbb{L}^{\circ}_{\mathrm{X}}.$$

Proof. Replace n by $2n$ in Lemma 2.4.1, and apply [Igu00, Theorem 8.3.1] again, we see

$$\int_{\mathrm{X}=\mathbb{V}_1} \Phi_0(v)\omega_1(v) = 1 - q^{-2n}.$$

On the other hand, according to [Tay92, Chapter 8], we have

$$\#\mathrm{Sp}_{2n}(\kappa_{\mathrm{F}}) = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1),$$

hence

$$\tau(\mathrm{X}) = \frac{\#\mathrm{X}(\kappa_{\mathrm{F}})}{q^{\dim \mathrm{X}}} = \frac{q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)}{q^{(n-1)^2}\prod_{i=1}^{n-1}(q^{2i}-1)}\frac{1}{q^{n(2n+1)-(n-1)(2n-1)}} = 1-q^{-2n}.$$

□

Lemma 2.6.2.

$$\overline{\Omega}(\Phi_0) = \frac{1}{\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(2n-2)}\Phi_{L\left(\mathrm{Std},\frac{n-1}{2}\right)\left(\mathrm{Std},\frac{n-3}{2}\right)}.$$

Proof. Using the Iwasawa decompositon of SL_2, write $g = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}m & 0\\ 0 & m^{-1}\end{pmatrix}k$ with $k \in \mathrm{SL}_2(\mathfrak{o}_{\mathrm{F}})$. But be careful that, according to our choices of measures, we have

$$\mathrm{d}g = \frac{1-q^{-2}}{1-q^{-1}}\,\mathrm{d}x|m|^{-2}\frac{\mathrm{d}m}{|m|}\,\mathrm{d}k,$$

where $\mathrm{d}k$ is the normalized measure such that

$$\int_{\mathrm{K}} 1\,\mathrm{d}k = 1.$$

Then we have

$$\begin{aligned}
&\overline{\Omega}(\Phi_0)\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix}\right)\\
&= \frac{1-q^{-2}}{1-q^{-1}}\int_{\mathrm{SL}_2}\Omega(\Phi_0)\left(\begin{pmatrix}\xi & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}m & \\ & m^{-1}\end{pmatrix}k\right)\,\mathrm{d}x|m|^{-2}\frac{\mathrm{d}m}{|m|}\,\mathrm{d}k\\
&= \frac{1-q^{-2}}{1-q^{-1}}\int_{1\le|m|\le|\xi|^{-1}}\int_{|x|\le|m\xi^{-1}|}\mathrm{d}x|m|^{-2}\frac{\mathrm{d}m}{|m|}\\
&= \frac{1-q^{-2}}{1-q^{-1}}|\xi|^{n-1}(1-q^{-1})\frac{1-q^{-1}|\xi|}{1-q^{-1}}\\
&= \frac{1}{\zeta_{\mathrm{F}}(2n-2)\zeta_{\mathrm{F}}(2)}\Phi_{L\left(\mathrm{Std},\frac{n-1}{2}\right)\left(\mathrm{Std},\frac{n-3}{2}\right)}.
\end{aligned}$$

□

Combine Lemma 2.6.1 and Lemma 2.6.2 together, we get

Theorem 2.6.3. *Let* $\mathbb{L}_{\mathrm{X}}^{\circ} \in \mathcal{S}(\mathrm{Sp}_{2n-2}\times\mathrm{SL}_2\backslash\mathrm{Sp}_{2n}/\mathrm{Sp}_{2n-2}\times\mathrm{Sp}_2)$ *be the push-forward of the probability measure which is equal to the characteristic function of* $\overline{\mathrm{X}}(\mathfrak{o}_{\mathrm{F}})\times\overline{\mathrm{X}}(\mathfrak{o}_{\mathrm{F}})$ *times an invariant measure, then we have*

$$\mathcal{T}^{-1}\left(\mathbb{L}_{\mathrm{X}}^{\circ}\right) = \frac{q^{\dim\overline{\mathrm{X}}}}{\#\overline{\mathrm{X}}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})}f_{L_{\mathrm{X}}} = \frac{\zeta_{\mathrm{F}}(2n)}{\zeta_{\mathrm{F}}(2)^2\zeta_{\mathrm{F}}(2n-2)}f_{L\left(\mathrm{Std},\frac{n-1}{2}\right)\left(\mathrm{Std},\frac{n-3}{2}\right)}.$$

2.6.2 Matching of Hecke elements

Let B be the Borel subgroup of Sp_{2n} fixing the isotropic flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \cdots, e_n\rangle,$$

and A the maximal torus of B fixing each line $\langle e_i\rangle$ for $1 \le i \le n$.

Lemma 2.6.4. *Let $\overline{\mathrm{X}}^{\circ}$ be the subset of $\overline{\mathrm{X}}$ consisting of 2-dimensional non-degenerate subspaces $\mathrm{V}_2 \subset \mathrm{F}^{2n}$ such that $\mathrm{V}_2 \cap \langle e_1, e_2\rangle^{\perp} = \{0\}$. Then $\overline{\mathrm{X}}^{\circ}$ is an open B-orbit in $\overline{\mathrm{X}}$.*

Proof. Let $\mathrm{V}_2 \in \overline{\mathrm{X}}^{\circ}$ and is generated by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

with

$$x_1 y_n + \cdots x_n y_{n+1} - x_{n+1} y_n - \cdots - x_{2n} y_1 = 1$$

under the ordered basis $\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$. As $\mathrm{V}_2 \cap \{e_1, e_2\}^{\perp} = \{0\}$, and

$$\begin{pmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{pmatrix}$$

is non-singular, we may assume

$$\begin{pmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{pmatrix} = \mathrm{I}_2.$$

Also from here one can see that $\overline{\mathrm{X}}^{\circ}$ is open in $\overline{\mathrm{X}}$.

First observe that

$$\begin{pmatrix} s_n \left({}^{t}\begin{pmatrix} 1 & & & -x_{n+1} & -y_{n+1} \\ & 1 & & -x_{n+2} & -y_{n+2} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix} \right)^{-1} s_n & 0_n \\ 0_n & \begin{pmatrix} 1 & & & -x_{n+1} & -y_{n+1} \\ & 1 & & -x_{n+2} & -y_{n+2} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} s_n \left({}^t\begin{pmatrix} 1 & & & -x_{n+1} & -y_{n+1} \\ & 1 & & -x_{n+2} & -y_{n+2} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix}^{-1} \right) s_n & 0_n \\ 0_n & \begin{pmatrix} 1 & & & -x_{n+1} & -y_{n+1} \\ & 1 & & -x_{n+2} & -y_{n+2} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

for some $x'_1, \cdots, x'_n, y'_1, \cdots, y'_n \in \mathrm{F}$. Then we can take

$$\begin{pmatrix} \mathrm{I}_n & \begin{pmatrix} -y'_n & -y'_{n-1} & \cdots & -y'_2 & -y'_1 \\ -x'_n & -x'_{n-1} & \cdots & -x'_2 & -y'_2 \\ 0 & 0 & \cdots & -x'_3 & -y'_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x'_{n-1} & -y'_{n-1} \\ 0 & 0 & 0 & -x'_n & -y'_n \end{pmatrix} \\ 0_n & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x'_1 - y'_2 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \mathrm{I}_n & \begin{pmatrix} -y'_n & -y'_{n-1} & \cdots & -y'_2 & -y'_1 \\ -x'_n & -x'_{n-1} & \cdots & -x'_2 & -y'_2 \\ 0 & 0 & \cdots & -x'_3 & -y'_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x'_{n-1} & -y'_{n-1} \\ 0 & 0 & 0 & -x'_n & -y'_n \end{pmatrix} \\ 0_n & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Moreover, according to the choice of the vectors at the very beginning, actually $x'_1 - y'_2 = 1$. Therefore $\overline{\mathrm{X}}^{\circ}$ is an B-orbit with $\langle e_1 + f_2, f_1 \rangle \in \overline{\mathrm{X}}^{\circ}$. $\square$

Corollary 2.6.5. $\mathrm{P}(\overline{\mathrm{X}})$ *is the parabolic subgroup fixing the isotropic flag*

$$\langle e_1, e_2 \rangle.$$

Then its Levi

$$\mathrm{L}(\overline{\mathrm{X}}) = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & s_2{}^t g^{-1} s_2 \end{pmatrix} \mid g \in \mathrm{GL}_2,\ h \in \mathrm{Sp}_{2n-4} \right\} \cong \mathrm{GL}_2 \times \mathrm{Sp}_{2n-2}$$

Therefore,

$$\delta^{\frac{1}{2}}_{(\overline{\mathrm{X}})} \left(\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \right) = |t_1|^{\frac{1}{2}} |t_2|^{-\frac{1}{2}} |t_3|^{n-2} \cdots |t_n|^1.$$

Moreover, the quotient map $\mathrm{A} \to \mathrm{A}_{\overline{\mathrm{X}}} \cong \mathbb{G}_m$ *is given by*

$$\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto t_1 t_2,$$

then after the Satake transform, the normalized map $\mathbb{C}[\mathrm{A}^\vee]^{\mathbb{W}_\mathrm{G}} \to \mathbb{C}[\mathrm{A}^\vee_{\overline{\mathrm{X}}}]^{\mathbb{W}_{\overline{\mathrm{X}}}}$ *is given by*

$$\mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathbb{W}_{\mathrm{Sp}_{2n}}} \to \mathbb{C}[x^{\pm}]^{\mathbb{Z}/2\mathbb{Z}} : x_1, x_2 \mapsto x,\ x_j \mapsto q^{j-n-1} \text{ for } 3 \le j \le n. \tag{2.10}$$

Write $\Sigma := \{v \in \mathbb{V} \mid \mathbb{B}_{\mathbb{V}}(v, v) = 0\}$ for the nilpotent cone.

Lemma 2.6.6.

$$\Sigma = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_{2n} \end{pmatrix} \otimes (1,0) + \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} \otimes (0,1) \mid (v_1, \cdots, v_{2n}) \cdot \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_{2n} \end{pmatrix} = 0 \right\},$$

and Σ *admits an open* $\mathrm{Sp}_{2n}(\mathrm{F}) \times \mathrm{SL}_2(\mathrm{F})$*-orbit* Σ°.

Proof. The first statement is clear. As for the second assertion, note that for any orthogonal non-zero $v, w \in \mathrm{F}^{2n}$, as long as they are not parallel to each other, there is some $g \in \mathrm{Sp}_{2n}$ such that $ge_1 = v$ and $ge_2 = w$. $\square$

Lemma 2.6.7. *Let $\Phi \in \mathcal{F}(\mathbb{Y})^{\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})}$, and*

$$\int_{\mathrm{SL}_2} \omega_\psi(g)\omega_\psi(x)\Phi(v)\,\mathrm{d}x = 0,\ \forall v \in \Sigma^\circ,$$

for all $g \in \mathrm{PGL}_2(\mathrm{F})$, then $\int_{\mathrm{SL}_2} \omega_\psi(x)\Phi(v) = 0$ for $v \in \mathbb{V}^\circ$, where $\mathbb{V}^\circ$ is the set of $v \otimes (1,0) + w \otimes (0,1)$ such that v and w are not parallel.

Proof. Consider the PGL_2-equivariant map

$$\mathcal{F}(\mathbb{Y}) \to \mathcal{C}^\infty(\mathbb{V}^\circ) : \Phi \mapsto \left(v \mapsto \int_{\mathrm{SL}_2} \omega_\psi(g)\Phi(v)\,\mathrm{d}g\right).$$

Let $\mathfrak{V}$ be the image of $\mathcal{F}(\mathbb{Y})$. Note that for any $\begin{pmatrix}1 & n\\ & 1\end{pmatrix} \in \mathrm{PGL}_2$, similarly for any $\Phi \in \mathcal{C}^\infty(\mathbb{V}^\circ)$,

$$\omega_\psi\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix}\right)\Phi(v) - \Phi(v) = \psi\left(\frac{n \cdot \tau(\mathbb{B}_\mathbb{V}(v,v))}{2}\right)\Phi(v),$$

the dual of the Jacquet module of $\mathcal{C}^\infty(\mathbb{V}^\circ)$ is the space of compact support distributions supported on $\mathbb{V}^\sigma \cap \Sigma = \Sigma^\circ$. As the Jacquet functor is exact, restriction to Σ° is the Jacquet functor of $\mathfrak{V}$. Therefore, according to the condition, the PGL_2-module generated by Φ inside $\mathfrak{V}$ is cuspidal, hence the only $\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})$-invariant element is 0. $\square$

Now let $\sigma \in \mathbb{C}$ be such that $\mathrm{Re}(\sigma) > 1$, and consider the intertwining operators

$$\begin{aligned}\mathrm{Z}_\sigma &: \mathcal{F}(\mathbb{Y}) \to \mathcal{C}^\infty(\mathrm{Sp}_{2n} \times \mathrm{PGL}_2)\\ &\Phi \mapsto \left((g_1, g_2) \mapsto \int_{\mathrm{SL}_2 \times \mathrm{F}} \omega_\psi(g_1, g_2)^{-1}\Phi((ae_1 \otimes (1,0) + e_2 \otimes (0,1))x)\,\mathrm{d}x|a|^\sigma \frac{\mathrm{d}a}{|a|}\right),\end{aligned}$$

then

Proposition 2.6.8. *When* $\mathrm{Re}(\sigma) > 1$, *(1) Let* $\mathfrak{s} = (s_1, s_2, \cdots, s_n) \in X^*(\mathrm{A}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^n$*, write*

$$\chi(\mathfrak{s}) : \mathrm{A} \to \mathbb{C} : \begin{pmatrix} t_1 &&&&& \\ & \ddots &&&& \\ && t_n &&& \\ &&& t_n^{-1} && \\ &&&& \ddots & \\ &&&&& t_1^{-1} \end{pmatrix} \mapsto \prod_{i=1}^{n} |t_i|^{s_i}$$

and $\mathrm{I}(\chi(\mathfrak{s}))$ *the correponding normalized induced representation of* Sp_{2n}*. Then* Z_σ *factors through* $\mathrm{I}(\chi(n-\sigma, n-1-\sigma, n-2, \cdots, 1)) \otimes \mathrm{I}(\chi(\sigma - n + \frac{1}{2}))$ *as representations of* $\mathrm{Sp}_{2n} \times \mathrm{PGL}_2$*.*

(2) Write K *for the maximal compact subgroup* $\mathrm{Sp}_{2n}(\mathfrak{o}_{\mathrm{F}})$*. Let*

$$\lambda_{\mathrm{X}} : \mathcal{H}(\mathrm{Sp}_{2n}, \mathrm{K}) \to \mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}}))$$

be the morphism of the Hecke algebra corresponding to (2.10). Let Φ_0 *be the characteristic function of* $\mathbb{Y}(\mathfrak{o}_{\mathrm{F}})$*, then we have*

$$\int_{\mathrm{SL}_2} (\omega_\psi(h^\vee) - \omega_\psi(\lambda_{\mathrm{X}}(h)))\omega_\psi(x)\Phi_0(v) \, \mathrm{d}v = 0.$$

for all $v \in \mathbb{V}^\circ$ *and* $h \in \mathcal{H}(\mathrm{Sp}_{2n}, \mathrm{K})$*.*

Proof. (1) follows immediately from the computation, and (2) is analogous to the arguments presented in the previous sections. □

Theorem 2.6.9. *Conjecture 2.1.5 is true for* X *of type* C_n*.*

Proof. Consider the function $\overline{\Omega}(\Phi_0)$ on PGL_2. As noted before,

$$\int_{\mathrm{F}} \overline{\Omega}(\Phi_0) \left(\mathbf{w} \begin{pmatrix} \xi & \\ & 1 \end{pmatrix} n \right) \psi^{-1}(n) \, \mathrm{d}n = \int_{\mathrm{SL}_2} \widehat{\mathbb{T}}(\mathcal{I} \circ \mathrm{res}(\Phi_0))(\xi, g) \, \mathrm{d}g.$$

Then the proof of the theorem follows by an argument analogous to that employed in the previous cases. □

2.7 Fundamental lemma of type B_n

In this section, we present two distinct approaches to proving the fundamental lemma. The first is a direct computational method, where we demonstrate that

$$\widetilde{\mathbb{T}}(\mathbb{L}_{\mathrm{X}}^{\circ}) = \frac{q^{\dim \mathrm{X}}}{\#\mathrm{X}(\kappa_{\mathrm{F}})\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(s_{\mathrm{X}})}\widetilde{\mathbb{T}} \circ \mathcal{T}\left(f_{L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)}\right)$$

without recourse to the metaplectic cover Mp_2. We establish this identity for the unit element in the initial section; while the computation is involved, this method offers a potential path for a generalization to arbitrary Hecke elements.

The second approach utilizes a diagrammatic argument to establish the commutativity of the diagram below. This commutativity, in conjunction with the fundamental lemmas for $\mathbb{T}$, $\overline{\mathrm{U}}$, and $\mathbb{J}$, provides a structural proof of the fundamental lemma for the transfer operator $\mathcal{T}$. This aligns the explicit computations required for our first approach with the computations in [Jac87].

$$\begin{tikzcd}
\mathcal{S}(\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}) \arrow[rr, "\widetilde{\mathbb{T}}"] & & \mathcal{S}_{\mathbb{V}}(\mathrm{T}) \\
& \mathcal{S}^{-}_{L\left(\mathrm{st},n-\frac{1}{2}\right)L\left(\mathrm{st},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{PGL}_2/\mathrm{N},\psi) \arrow[ul, "\mathcal{T}"] \arrow[ur, "\widetilde{\mathbb{T}}\circ\mathcal{T}"'] & \\
& \mathcal{S}^{-}_{L\left(\mathrm{st},n-\frac{1}{2}\right)}(\mathrm{A}\backslash \mathrm{PGL}_2/\mathrm{N},\psi) \arrow[u, "\overline{\mathrm{U}}"] \arrow[uur, dotted, bend right, "\mathbb{J}"'] &
\end{tikzcd} \tag{2.11}$$

We provide a brief overview of the operators appearing in the diagram above.

- $\widetilde{\mathbb{T}}$ denotes the transfer operator arising from the Weil representation,
- $\mathcal{T}$ is the transfer operator defined in [Sak21],
- $\overline{\mathrm{U}}$ corresponds to the unfolding process, but with non-standard test measures to ensure compatibility with the space of orbital integrals,
- $\mathbb{J}$ represents the comparison established by H. Jacquet in [Jac87], and we extend this comparison from standard test measures to the non-standard unit element required for our current setting.

Section 2.7.1 is devoted to the direct computational proof of the fundamental lemma. The subsequent sections develop the structure proof by establishing the commutativity of the diagram relating the transfer operators, the Weil representations, and the Jacquet comparison.

In this section, let $\mathbb{D} = \mathrm{F}$, $\dim_{\mathrm{F}} \mathbb{V}$ is odd, suppose there is an ordered basis

$$\{e_1, \cdots, e_n, w, f_n, \cdots, f_1\}$$

such that

$$\mathbb{B}_{\mathbb{V}}(e_i, e_j) = \mathbb{B}_{\mathbb{V}}(f_i, f_j) = 0, \text{ and } \mathbb{B}_{\mathbb{V}}(e_i, f_j) = \delta_{ij} \ \forall\, 1 \leq i, j \leq n,$$

$$\mathbb{B}_{\mathbb{V}}(e_i, w) = \mathbb{B}_{\mathbb{V}}(f_i, w) = 0, \ \forall\, 1 \leq i \leq n,$$

and

$$\mathbb{B}_{\mathbb{V}}(w, w) = 2.$$

Under the isomorphism $\mathbb{V} \cong \mathrm{F}^{2n+1}$ under the above basis, we have

$$\mathbb{B}_{\mathbb{V}} : \mathrm{F}^{2n+1} \times \mathrm{F}^{2n+1} \to \mathrm{F} : \left(\begin{pmatrix} x_1 \\ \vdots \\ x_{2n+1} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2n+1} \end{pmatrix} \right) \mapsto \sum_{i=1}^{2n+1} x_i y_{2n+2-i} + x_{n+1} y_{n+1}.$$

We take $\mathbb{W}$ to be a 2-dimensional symplectic space over F, and fix the ordered basis $\{e, f\}$ of $\mathbb{W}$ such that $\mathbb{B}_{\mathbb{W}}(e, f) = 1$. Under the isomorphism $\mathbb{W} \cong \mathrm{F}^2$ under this fixed basis, we have

$$\mathbb{B}_{\mathbb{W}} : \mathrm{F}^2 \times \mathrm{F}^2 \to \mathrm{F} : \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto x_1 y_2 - x_2 y_1.$$

In this case, $\mathrm{U}(\mathbb{V}, \mathbb{B}_{\mathbb{V}}) = \mathrm{SO}(\mathbb{V})$ and $\mathrm{U}(\mathbb{W}, \mathbb{B}_{\mathbb{W}}) = \mathrm{SL}_2$. Moreover, by the composition

$$\mathrm{Mp}_2 \to \widetilde{\mathrm{SL}_2} \to \mathrm{Mp}(\mathbb{V} \otimes_{\mathbb{D}} \mathbb{W})^{\mathbb{C}^1}_{\psi, \mathbb{X}},$$

we obtain a Weil representation of Mp_2. Let Φ_0 be the characteristic function of $\mathbb{Y}(\mathfrak{o}_{\mathrm{F}})$.

2.7.1 Matching of unit elements

To compare $f_{L(\mathrm{Std},\mathrm{n}-\frac{1}{2})L(\mathrm{Std},\frac{1}{2})}$ with $\mathbb{L}_{\mathrm{X}}^{\circ}$, our strategy is to compute

$$\widetilde{\mathbb{T}}\circ\mathcal{T}\left(f_{L(\mathrm{Std},\mathrm{n}-\frac{1}{2})L(\mathrm{Std},\frac{1}{2})}\right) \text{ and } \widetilde{\mathbb{T}}(\mathbb{L}_{\mathrm{X}}^{\circ})$$

separately. Let us start with computing $\widetilde{\mathbb{T}}(\mathbb{L}_{\mathrm{X}}^{\circ})$.

Lemma 2.7.1. *There is some k large enough, such that*

(1) when $|\zeta|\geq 1$,

$$\begin{aligned}&\int_{\mathrm{N}}^{*}\omega_{\psi}\left(\mathbf{w}\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}u\right)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u\\ &=\gamma(\zeta,\psi)|\zeta|^{-\frac{2n+1}{2}}\left(\int_{|u|\leq 1}\psi^{-1}(u)\int_{z\in\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u+\int_{1<|u|\leq q^k}\psi^{-1}(u)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u\right).\end{aligned}$$

(2) when $|\zeta|<1$,

$$\begin{aligned}&\int_{\mathrm{N}}^{*}\omega_{\psi}\left(\mathbf{w}\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}u\right)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u\\ &=\gamma(\zeta,\psi)|\zeta|^{-\frac{2n+1}{2}}\int_{|\zeta|^{-1}\leq|u|\leq q^k}\psi^{-1}\left(\frac{1}{u\zeta^2}+u\right)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u.\end{aligned}$$

Proof.

$$\begin{aligned}&\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\omega_{\psi}\left(\mathbf{w}\begin{pmatrix}\zeta & \\ & \zeta^{-1}\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u\\
&=\gamma(\zeta,\psi)\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathrm{F}^{2n+1}}\omega_{\psi}\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\Phi_0((x_1,\cdots,x_n,z,y_1,\cdots,y_n))\\
&\cdot\psi^{-1}(x_n+y_n)\,\mathrm{d}x_1\cdots\mathrm{d}y_n\,\mathrm{d}z\cdot\psi^{-1}(u)\,\mathrm{d}u\\
&=\gamma(\zeta,\psi)\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathrm{F}^{2n+1}}|\zeta|^{\frac{2n+1}{2}}\psi(u\zeta^2(x_1y_1+\cdots x_ny_n+z^2))\Phi_0((\zeta x_1,\cdots,\zeta x_n,\zeta z,\zeta y_1,\cdots,\zeta y_n))\\
&\cdot\psi^{-1}(x_n+y_n)\,\mathrm{d}x_1\cdots\mathrm{d}y_n\,\mathrm{d}z\cdot\psi^{-1}(u)\,\mathrm{d}u\\
&=\gamma(\zeta,\psi)\int_{\mathfrak{p}_{\mathrm{F}}^{-k}}\int_{\mathrm{F}^{2n+1}}|\zeta|^{-\frac{2n+1}{2}}\psi(u(x_1y_1+\cdots x_ny_n+z^2))\Phi_0((x_1,\cdots,x_n,z,y_1,\cdots,y_n))\\
&\cdot\psi^{-1}(\zeta^{-1}x_n+\zeta^{-1}y_n)\,\mathrm{d}x_1\cdots\mathrm{d}y_n\cdot\psi^{-1}(u)\,\mathrm{d}u.\end{aligned}$$

Note that we have

$$\int_{\mathrm{F}^n}\Phi_0((x_1,\cdots,x_n,z,y_1,\cdots,y_n))\psi(u(x_1y_1+\cdots x_ny_n))\psi^{-1}(\zeta^{-1}x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$=\Phi_0(uy_1,\cdots,uy_{n-1},uy_n-\zeta^{-1},z,y_1,\cdots,y_n),$$

then integrate over the other F^n, we obtain

$$\int_{\mathrm{F}^n}\Phi_0(uy_1,\cdots,uy_{n-1},uy_n-\zeta^{-1},z,y_1,\cdots,y_n)\psi^{-1}(\zeta^{-1}y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$$
$$=1_{\mathfrak{o}_\mathrm{F}}(z)\int_{\mathrm{F}^{n-1}}1_{\mathfrak{o}_\mathrm{F}^{n-1}}(uy_1,\cdots,uy_{n-1})1_{\mathfrak{o}_\mathrm{F}^{n-1}}(y_1,\cdots,y_{n-1})\,\mathrm{d}y_1\cdots\mathrm{d}y_{n-1}$$
$$\cdot\int_{\mathrm{F}}1_{\mathfrak{o}_\mathrm{F}}(uy_n-\zeta^{-1})1_{\mathfrak{o}_\mathrm{F}}(y_n)\psi^{-1}(\zeta^{-1}y_n)\,\mathrm{d}y_n.$$

It is easy to see that

$$\int_{\mathrm{F}}1_{\mathfrak{o}_\mathrm{F}}(uy)1_{\mathfrak{o}_\mathrm{F}}(y)\,\mathrm{d}y=\begin{cases}1 & \text{for}\quad |u|\le 1;\\ |u|^{-1} & \text{for}\quad |u|>1.\end{cases}$$

Moreover, when $u\neq 0$,

$$\int_{\mathrm{F}}1_{\mathfrak{o}_\mathrm{F}}(uy-\zeta^{-1})1_{\mathfrak{o}_\mathrm{F}}(y)\psi^{-1}(\zeta^{-1}y)\,\mathrm{d}y$$
$$=\int_{\mathrm{F}}1_{\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})}(y)\psi^{-1}(t_2y)\,\mathrm{d}y$$
$$=\mathbb{F}_\psi\left(1_{\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})}\right)(\zeta^{-1}),$$

where $1_{\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})}$ is the characteristic function of $\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})$. Note that

$$\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})=\begin{cases}\mathfrak{o}_\mathrm{F} & \text{for}\quad |u|\le 1,\ |\zeta|\ge 1;\\ \emptyset & \text{for}\quad |u|\le 1,\ |\zeta|<1;\\ \zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F} & \text{for}\quad |u|>1,\ |\zeta|\ge|u|^{-1};\\ \emptyset & \text{for}\quad |u|>1,\ |\zeta|<|u|^{-1}.\end{cases}$$

Then we have its Fourier transform

$$\mathbb{F}_\psi\left(1_{\mathfrak{o}_\mathrm{F}\cap(\zeta^{-1}u^{-1}+u^{-1}\mathfrak{o}_\mathrm{F})}\right)(\zeta^{-1})=\begin{cases}1 & \text{for}\quad |u|\le 1,\ |\zeta|\ge 1;\\ 0 & \text{for}\quad |u|\le 1,\ |\zeta|<1;\\ \psi(-\zeta^{-2}u^{-1})|u|^{-1}1_{u\mathfrak{o}_\mathrm{F}}(\zeta^{-1}) & \text{for}\quad |u|>1,\ |\zeta|\ge|u|^{-1};\\ 0 & \text{for}\quad |u|>1,\ |\zeta|<|u|^{-1}.\end{cases}$$

Therefore, when $|u| \leq 1$, we have the integrand function is

$$\begin{cases} |\zeta|^{-\frac{2n+1}{2}}\psi^{-1}(u) & \text{for } |\zeta| \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

And when $|u| > 1$, the integrand function is

$$\begin{cases} |\zeta|^{-\frac{2n+1}{2}}\psi^{-1}\left(\frac{1}{u\zeta^2}+u\right)|u|^{-n} & \text{for } u \geq \max\{q, |\zeta|^{-1}\} \\ 0 & \text{otherwise.} \end{cases}$$

Then, when k is large enough, the final integral becomes:

(1) when $|\zeta| \geq 1$,

$$\gamma(\zeta,\psi)|\zeta|^{-\frac{2n+1}{2}}\left(\int_{|u|\leq 1}\psi^{-1}(u)\int_{z\in\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u+\int_{1<|u|\leq q^k}\psi^{-1}(u)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u\right).$$

(2) when $|\zeta| < 1$,

$$\gamma(\zeta,\psi)|\zeta|^{-\frac{2n+1}{2}}\int_{|\zeta|^{-1}\leq|u|\leq q^k}\psi^{-1}\left(\frac{1}{u\zeta^2}+u\right)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u.$$

□

Lemma 2.7.2.

$$\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z=\begin{cases} 1 & \textit{for} \quad |u|\leq 1; \\ |u|^{-\frac{1}{2}} & \textit{for} \quad |u|>1,\ \log_q|u|\equiv 0 \mod 2; \\ q^{-\frac{1}{2}}|u|^{-\frac{1}{2}}\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(\frac{\mathrm{ac}(u)\cdot a^2}{\varpi_{\mathrm{F}}}\right) & \textit{for} \quad |u|>1,\ \log_q|u|\equiv 1 \mod 2. \end{cases}$$

Proof. The case that $|u| \leq 1$ is obvious, and we only have to consider the case that $|u| > 1$. In the case that $|u| = q^{2r}$ for some $r > 0$, we have

$$\begin{aligned}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^r}\int_{\mathfrak{p}_{\mathrm{F}}^r}\psi(u(a+z)^2)\,\mathrm{d}z \\ &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^r}\int_{\mathfrak{p}_{\mathrm{F}}^r}\psi(a^2u+2azu+z^2u)\,\mathrm{d}z \\ &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^r}\psi(a^2u)\int_{\mathfrak{p}_{\mathrm{F}}^r}\psi(2azu)\,\mathrm{d}z = q^{-r} = |u|^{-\frac{1}{2}}.\end{aligned}$$

If $|x| = q^{2r+1}$ for some $r \geq 0$, similarly we have

$$\begin{aligned}\int_{\mathfrak{o}_{\mathrm{F}}} \psi(uz^2)\,\mathrm{d}z &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{r+1}} \int_{\mathfrak{p}_{\mathrm{F}}^{r+1}} \psi(u(a+z)^2)\,\mathrm{d}z \\ &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{r+1}} \int_{\mathfrak{p}_{\mathrm{F}}^{r+1}} \psi(a^2u + 2azu + z^2u)\,\mathrm{d}z \\ &= \sum_{a\in\mathfrak{o}_{\mathrm{F}}/\mathfrak{p}_{\mathrm{F}}^{r+1}} \psi(a^2u) \int_{\mathfrak{p}_{\mathrm{F}}^{r+1}} \psi(2azu)\,\mathrm{d}z \\ &= \sum_{a\in\mathfrak{p}_{\mathrm{F}}^{r}/\mathfrak{p}_{\mathrm{F}}^{r+1}} \psi(a^2u)q^{-(r+1)} = q^{-\frac{1}{2}}|u|^{-\frac{1}{2}} \sum_{a\in\kappa_{\mathrm{F}}} \psi\left(\frac{ac(u)\cdot a^2}{\varpi_{\mathrm{F}}}\right).\end{aligned}$$

□

Lemma 2.7.3. *When $|\zeta| < 1$, we have*

$$\int_{\mathrm{N}}^{*} \omega_\psi(\mathbf{w})\omega_\psi\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right)\omega_\psi(u)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u \\ = \gamma(\zeta,\psi)|\zeta|^{-\frac{1}{2}}\left(\psi\left(\frac{2}{\zeta}\right)+\psi\left(\frac{-2}{\zeta}\right)\right).$$

Proof. Let us start with the integral

$$|\zeta|^{-\frac{2n+1}{2}} \int_{|x|=q^i} \psi^{-1}\left(\frac{1}{u\zeta^2}+u\right)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u$$

for some i such that $|\zeta|^{-1} \leq |u| \leq q^k$. In the case that i is even, this integral becomes

$$|\zeta|^{-\frac{2n+1}{2}} \int_{|u|=q^i} \psi^{-1}\left(\frac{1}{u\zeta^2}+u\right)|u|^{-n-\frac{1}{2}}\,\mathrm{d}u,$$

which is non-zero only when $|u| = |\zeta|^{-1}$. In this case, we may make a change of variable u to $\zeta^{-1}u$, then it becomes

$$|\zeta|^{-1}\int_{\mathfrak{o}_{\mathrm{F}}^{\times}} \psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right)\mathrm{d}u.$$

Write $|\zeta| = q^{-2r}$ with $r > 0$, then the above is

$$\begin{aligned}
&|\zeta|^{-1} \sum_{a \in \mathfrak{o}_{\mathrm{F}}^{\times}/(1+\mathfrak{p}_{\mathrm{F}}^{r})} \int_{\mathfrak{p}_{\mathrm{F}}^{r}} \psi^{-1}\left(\frac{a(1+z)+a^{-1}(1-z)}{\zeta}\right) \mathrm{d}z \\
&= |\zeta|^{-1} \cdot |\zeta|^{\frac{1}{2}} \left(\psi\left(\frac{2}{\zeta}\right) + \psi\left(\frac{-2}{\zeta}\right)\right) \\
&= |\zeta|^{-\frac{1}{2}} \left(\psi\left(\frac{2}{\zeta}\right) + \psi\left(\frac{-2}{\zeta}\right)\right).
\end{aligned}$$

When $|u| = q^i$ such that $i > 1$ is odd,

$$\begin{aligned}
&|\zeta|^{-\frac{2n+1}{2}} \int_{|u|=q^i} \psi^{-1}\left(\frac{1}{u\zeta^2} + u\right) |u|^{-n} \int_{\mathfrak{o}_{\mathrm{F}}} \psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u \\
&= |\zeta|^{-\frac{2n+1}{2}} \int_{|u|=q^i} |u|^{-n-\frac{1}{2}} q^{-\frac{1}{2}} \psi^{-1}\left(\frac{1}{u\zeta^2} + u\right) \sum_{a \in \kappa_{\mathrm{F}}} \psi\left(\frac{\mathrm{ac}(x) \cdot a^2}{\varpi_{\mathrm{F}}}\right) \mathrm{d}u.
\end{aligned}$$

Replace u by $\zeta^{-1}u$,

$$\begin{aligned}
&|\zeta|^{-\frac{2n+1}{2}} \int_{|u|=q^i} |u|^{-n-\frac{1}{2}} q^{-\frac{1}{2}} \psi^{-1}\left(\frac{1}{u\zeta^2} + u\right) \sum_{a \in \kappa_{\mathrm{F}}} \psi\left(\frac{\mathrm{ac}(x) \cdot a^2}{\varpi_{\mathrm{F}}}\right) \mathrm{d}u \\
&= q^{-\frac{1}{2}} |\zeta|^{-1} \int_{|u|=|\zeta|q^i \geq 1} \psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right) \sum_{a \in \kappa_{\mathrm{F}}} \psi\left(\frac{ac(u)a^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right) \mathrm{d}u.
\end{aligned}$$

Write $|\zeta|q^i = q^r$ for some $r \geq 0$ and $u = \varpi_{\mathrm{F}}^{-r} v$ with $v \in \mathfrak{o}_{\mathrm{F}}^{\times}$, then

$$\begin{aligned}
&q^{-\frac{1}{2}} |\zeta|^{-1} \int_{|u|=|\zeta|q^i \geq 1} \psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right) \sum_{a \in \kappa_{\mathrm{F}}} \psi\left(\frac{ac(u)a^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right) \mathrm{d}u \\
&= q^{i-\frac{1}{2}} \int_{v \in \mathfrak{o}_{\mathrm{F}}\times} \psi^{-1}\left(\frac{\varpi_{\mathrm{F}}^{-r} v + \varpi_{\mathrm{F}}^{r} v^{-1}}{\zeta}\right) \sum_{a \in \kappa_{\mathrm{F}}} \psi\left(\frac{va^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right) \mathrm{d}v \\
&= q^{i-\frac{1}{2}} \sum_{a \in \kappa_{\mathrm{F}}} \int_{v \in \mathfrak{o}_{\mathrm{F}}^{\times}} \psi^{-1}\left(\frac{\varpi_{\mathrm{F}}^{-r} v + \varpi_{\mathrm{F}}^{r} v^{-1}}{\zeta} - \frac{a^2 v}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right) \mathrm{d}v \\
&= q^{i-\frac{1}{2}} \sum_{a \in \kappa_{\mathrm{F}}} \int_{v \in \mathfrak{o}_{\mathrm{F}}^{\times}} \psi^{-1}\left(\frac{1}{ac(\zeta)}\left(\left(\frac{1}{\varpi_{\mathrm{F}}^{i}} - \frac{a^2}{\varpi_{\mathrm{F}}}\right) v + \frac{\varpi_{\mathrm{F}}^{2r}}{\varpi_{\mathrm{F}}^{i}} v^{-1}\right)\right) \mathrm{d}v \\
&= q^{i-\frac{1}{2}} \sum_{a \in \kappa_{\mathrm{F}}} \int_{v \in \mathfrak{o}_{\mathrm{F}}^{\times}} \psi^{-1}\left(\frac{1}{ac(\zeta)}\left(\frac{1-\varpi_{\mathrm{F}}^{i-1} a^2}{\varpi_{\mathrm{F}}^{i}} v + \frac{\varpi_{\mathrm{F}}^{2r}}{\varpi_{\mathrm{F}}^{i}} v^{-1}\right)\right) \mathrm{d}v
\end{aligned}$$

Write $i = 2m+1$ with $m \geq 1$, $v = b(1+w)$ with $b \in \mathfrak{o}_{\mathrm{F}}^{\times}/(1+\mathfrak{p}_{\mathrm{F}}^{m+1})$ and $w \in \mathfrak{p}_{\mathrm{F}}^{m+1}$. For the integral being non-zero, we need

$$(1-\varpi_{\mathrm{F}}^{2m} a^2) b - \varpi_{\mathrm{F}}^{2r} b^{-1} \in \mathfrak{p}_{\mathrm{F}}^{m} \Leftrightarrow b - \varpi_{\mathrm{F}}^{2r} b^{-1} \in \mathfrak{p}_{\mathrm{F}}^{m}.$$

The only possibility is $r=0$, and we must have $b \equiv b^{-1} \mod \mathfrak{p}_{\mathrm{F}}^m$, in which case

$$q^{i-\frac{1}{2}} \sum_{a\in\kappa_{\mathrm{F}}} \int_{v\in\mathfrak{o}_{\mathrm{F}}^\times} \psi^{-1}\left(\frac{1}{ac(\zeta)}\left(\frac{1-\varpi_{\mathrm{F}}^{i-1}a^2}{\varpi_{\mathrm{F}}^i}v+\frac{\varpi_{\mathrm{F}}^{2r}}{\varpi_{\mathrm{F}}^i}v^{-1}\right)\right) \mathrm{d}v$$
$$= q^{i-\frac{1}{2}}q^{-m-1}\sum_{a\in\kappa_{\mathrm{F}}}\left\{\sum_{1+\mathfrak{p}_{\mathrm{F}}^m/1+\mathfrak{p}_{\mathrm{F}}^{m+1}} + \sum_{-1+\mathfrak{p}_{\mathrm{F}}^m/1+\mathfrak{p}_{\mathrm{F}}^{m+1}}\right\}\psi^{-1}\left(\frac{1}{\mathrm{ac}(\zeta)}\left(\frac{b+b^{-1}}{\varpi_{\mathrm{F}}^{2m+1}}-\frac{a^2b}{\varpi_{\mathrm{F}}}\right)\right).$$

As $m\geq 1$, the summation over a is independent of b, hence it is

$$q^{-1}|\zeta|^{-\frac{1}{2}}\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(\frac{a^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)\sum_{1+\mathfrak{p}_{\mathrm{F}}^m/1+\mathfrak{p}_{\mathrm{F}}^{m+1}}\psi^{-1}\left(\frac{b+b^{-1}}{\zeta}\right)+$$
$$q^{-1}|\zeta|^{-\frac{1}{2}}\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(-\frac{a^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)\sum_{-1+\mathfrak{p}_{\mathrm{F}}^m/1+\mathfrak{p}_{\mathrm{F}}^{m+1}}\psi^{-1}\left(\frac{b+b^{-1}}{\zeta}\right).$$

Write $b=1+u$ with $u\in\mathfrak{p}_{\mathrm{F}}^m/\mathfrak{p}_{\mathrm{F}}^{m+1}$, then we have

$$\sum_{b\in 1+\mathfrak{p}_{\mathrm{F}}^m/1+\mathfrak{p}_{\mathrm{F}}^{m+1}}\psi^{-1}\left(\frac{b+b^{-1}}{\zeta}\right)=\psi^{-1}\left(\frac{2}{\zeta}\right)\sum_{u\in\mathfrak{p}_{\mathrm{F}}^m/\mathfrak{p}_{\mathrm{F}}^{m+1}}\psi^{-1}\left(\frac{u^2}{\zeta}\right)$$
$$=\psi^{-1}\left(\frac{2}{\zeta}\right)\sum_{u\in\kappa_{\mathrm{F}}}\psi^{-1}\left(\frac{u^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right).$$

It is clear that

$$\sum_{u\in\kappa_{\mathrm{F}}}\psi^{-1}\left(\frac{u^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(\frac{a^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)=q,$$

hence, the original integral is still

$$|\zeta|^{-\frac{1}{2}}\left(\psi\left(\frac{2}{\zeta}\right)+\psi\left(\frac{-2}{\zeta}\right)\right).$$

The last case is when $|\zeta|=q^{-1}$. When $i\geq 3$, we have $r=i-1$. At this time, we have

$$\left|\frac{\varpi_{\mathrm{F}}^{2r}}{\varpi_{\mathrm{F}}^i}v^{-1}\right|\leq 1,$$

and

$$\left|\frac{1-\varpi_{\mathrm{F}}^{i-1}a^2}{\varpi_{\mathrm{F}}^i}\right|>q,$$

hence

$$q^{i-\frac{1}{2}}\sum_{a\in\kappa_{\mathrm{F}}}\int_{v\in\mathfrak{o}_{\mathrm{F}}^{\times}}\psi^{-1}\left(\frac{1}{ac(\zeta)}\left(\frac{1-\varpi_{\mathrm{F}}^{i-1}a^2}{\varpi_{\mathrm{F}}^{i}}v+\frac{\varpi_{\mathrm{F}}^{2r}}{\varpi_{\mathrm{F}}^{i}}v^{-1}\right)\right)\mathrm{d}v=0.$$

So the only case left is when $i=1$. Make a change of variable that $x=\zeta^{-1}u$, we have

$$\begin{aligned}&|\zeta|^{-\frac{2n+1}{2}}\int_{|x|=|\zeta|^{-1}}\psi^{-1}\left(\frac{1}{x\zeta^2}+x\right)|x|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(xz^2)\,\mathrm{d}z\\&=q^{-\frac{1}{2}}|\zeta|^{-1}\int_{x\in\mathfrak{o}_{\mathrm{F}}\times}\psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right)\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(\frac{ua^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)\mathrm{d}u.\end{aligned}$$

Write $u=b(1+v)$ with $b\in\mathfrak{o}_{\mathrm{F}}^{\times}/1+\mathfrak{p}_{\mathrm{F}}$ and $v\in\mathfrak{p}_{\mathrm{F}}$, then we have

$$\begin{aligned}\psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right)\psi\left(\frac{ua^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)&=\psi^{-1}\left(\frac{b(1+v)+b^{-1}(1-v)-b(1+v)a^2}{\zeta}\right)\\&=\psi^{-1}\left(\frac{b+b^{-1}-ba^2}{\zeta}\right)\cdot\psi^{-1}\left(\frac{b-b^{-1}-ba^2}{\zeta}\cdot v\right)\\&=\psi^{-1}\left(\frac{b+b^{-1}-ba^2}{\zeta}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&q^{-\frac{1}{2}}|\zeta|^{-1}\int_{x\in\mathfrak{o}_{\mathrm{F}}\times}\psi^{-1}\left(\frac{u+u^{-1}}{\zeta}\right)\sum_{a\in\kappa_{\mathrm{F}}}\psi\left(\frac{ua^2}{\mathrm{ac}(\zeta)\varpi_{\mathrm{F}}}\right)\mathrm{d}u\\&=q^{-\frac{1}{2}}|\zeta|^{-1}q^{-1}\sum_{b\in\mathfrak{o}_{\mathrm{F}}^{\times}/1+\mathfrak{p}_{\mathrm{F}}}\sum_{a\in\kappa_{\mathrm{F}}}\psi^{-1}\left(\frac{b+b^{-1}-ba^2}{\zeta}\right)\end{aligned}$$

Then we claim first for $\alpha=\pm2$,

$$\#\{(a\in\kappa_{\mathrm{F}},b\in\kappa_{\mathrm{F}}^{\times})\mid b+b^{-1}-ba^2=\alpha\}=2q-3.$$

Indeed, for $\alpha=2$, this is equivalent to count the pairs (a,b) such that $a^2=(1-b^{-1})^2$, so there are totally $2(q-1-1)+1=2q-3$ pairs. Similarly for$\alpha\neq2$.

For $\alpha\neq\pm2$, we claim

$$\#\{(a\in\kappa_{\mathrm{F}},b\in\kappa_{\mathrm{F}}^{\times})\mid b+b^{-1}-ba^2=\alpha\}=q-3.$$

In fact, when α^2-4 is a square, it is known that there are $\frac{q+1}{2}$ elements in κ_{F} such that $z^2-\alpha z+1$ is a square, hence ther number of pairs (a,b) is

$$2\left(\frac{q+1}{2}-1-2\right)+2=q-3.$$

When α^2-4 is not a square, it is also known that there are $\dfrac{q-1}{2}$ elements in κ_{F} such that $z^2-\alpha z+1$ is a square, hence, the total number is

$$2\left(\frac{q-1}{2}-1\right)=q-3.$$

Therefore,

$$q^{-\frac{1}{2}}|\zeta|^{-1}q^{-1}\sum_{b\in\mathfrak{o}_{\mathrm{F}}^{\times}/1+\mathfrak{p}_{\mathrm{F}}}\sum_{a\in\kappa_{\mathrm{F}}}\psi^{-1}\left(\frac{b+b^{-1}-ba^2}{\zeta}\right)$$
$$=|\zeta|^{-\frac{1}{2}}\left(\psi\left(\frac{2}{\zeta}\right)+\psi\left(\frac{-2}{\zeta}\right)\right).$$

□

Lemma 2.7.4. *When $|\zeta|\geq 1$, we have*

$$\int_{\mathrm{N}}^{*}\omega_{\psi}(\mathbf{w})\omega_{\psi}\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right)\omega_{\psi}(u)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u=\gamma(\zeta,\psi)|\zeta|^{-n-\frac{1}{2}}(1+q^{-n}).$$

Proof. When $|\zeta|\geq 1$, according to the above Lemma 2.7.1, there is some k large enough such that we have

$$\int_{\mathrm{N}}^{*}\omega_{\psi}(\mathbf{w})\omega_{\psi}\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right)\omega_{\psi}(u)\Phi_0(v_0)\psi^{-1}(u)\,\mathrm{d}u$$
$$=\gamma(\zeta,\psi)|\zeta|^{-\frac{2n+1}{2}}\left(\int_{|u|\leq 1}\psi^{-1}(u)\int_{z\in\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u+\int_{1<|u|\leq q^k}\psi^{-1}(u)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u\right).$$

It is clear that

$$\int_{|u|\leq 1}\psi^{-1}(u)\int_{z\in\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u=1.$$

Now, let us consider the integral

$$\int_{|u|=q^i}\psi^{-1}(u)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u,$$

where $0<i\leq k$. If i is even, we know

$$\int_{|u|=q^i}\psi^{-1}(u)|u|^{-n}\int_{\mathfrak{o}_{\mathrm{F}}}\psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u=0$$

due to Lemma 2.7.2. When i is odd, according to Lemma 2.7.2 again, we have

$$\begin{aligned}
&\int_{|u|=q^i} \psi^{-1}(u)|u|^{-n} \int_{\mathfrak{o}_\mathrm{F}} \psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u \\
&= \int_{|u|=q^i} \psi^{-1}(u)|u|^{-n} q^{-\frac{1}{2}}|u|^{-\frac{1}{2}} \sum_{a\in\kappa_\mathrm{F}} \psi\left(\frac{\mathrm{ac}(u)\cdot a^2}{\varpi_\mathrm{F}}\right)\,\mathrm{d}u \\
&= q^{-i(n+\frac{1}{2})-\frac{1}{2}} \sum_{a\in\kappa_\mathrm{F}} \int_{|u|=q^i} \psi^{-1}\left(u\left(1-a^2\varpi_\mathrm{F}^{i-1}\right)\right)\,\mathrm{d}u.
\end{aligned}$$

If $i \geq 3$, $|1-a^2\varpi_\mathrm{F}^{i-1}| = 1$, then

$$\int_{|u|=q^i} \psi^{-1}\left(u\left(1-a^2\varpi_\mathrm{F}^{i-1}\right)\right)\,\mathrm{d}u = 0$$

for all $a \in \kappa_\mathrm{F}$. If $i = 1$, then

$$\begin{aligned}
&\int_{|u|=q^i} \psi^{-1}(u)|u|^{-n} \int_{\mathfrak{o}_\mathrm{F}} \psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u \\
&= q^{-n-1} \sum_{a\in\kappa_\mathrm{F}} \int_{|u|=q} \psi^{-1}(u(1-a^2))\,\mathrm{d}u \\
&= q^{-n-1} \left\{ \sum_{a=\pm 1} + \sum_{a\in\kappa_\mathrm{F}\setminus\{\pm 1\}} \right\} \int_{|u|=q} \psi^{-1}(u(1-a^2))\,\mathrm{d}u \\
&= q^{-n-1}(2(q-1)+(q-2)(-1)) \\
&= q^{-n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\zeta|^{-\frac{2n+1}{2}} \left(\int_{|u|\leq 1} \psi^{-1}(u) \int_{z\in\mathfrak{o}_\mathrm{F}} \psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u + \int_{1<|u|\leq q^k} \psi^{-1}(u)|u|^{-n} \int_{\mathfrak{o}_\mathrm{F}} \psi(uz^2)\,\mathrm{d}z\,\mathrm{d}u \right) \\
&= |\zeta|^{-n-\frac{1}{2}}(1+q^{-n}).
\end{aligned}$$

$\square$

Combine Lemma 2.7.3 and Lemma 2.7.4, we have

Proposition 2.7.5.

$$\widetilde{\mathbb{T}}(\mathcal{I}(\mathrm{res}(\Phi_0)))\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right) = \gamma(\zeta,\psi)\begin{cases} |\zeta|^{-\frac{1}{2}}\left(\psi^{-1}\left(\frac{2}{\zeta}\right)+\psi\left(\frac{2}{\zeta}\right)\right) & \textit{for} \quad |\zeta|\leq q^{-1};\\ |\zeta|^{-n-\frac{1}{2}}(1+q^{-n}) & \textit{for} \quad |\zeta|\geq 1.\end{cases}$$

Next let us compute $\widetilde{\mathbb{T}} \circ \mathcal{T}\left(f_{L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)}\right)$. It is clear that

$$\iota \circ |\cdot| \left(f_{L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)}\left(\left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix}\right)\right)\right)(\xi) = (1-q^{-2})^{-1}(1-q^{n-1})^{-1}\left(1-q^{-n}\right)^{-1}$$
$$\cdot \begin{cases} |\xi|^{n-1}(1-q^{-2n}) - q^{n-1}(1-q^{-2}) & \text{for} \quad |\xi| \leq 1; \\ -q^{-n-1} + q^{n-3} & \text{for} \quad |\xi| = q; \\ (1-q^{n-1}) \int_{|x|^2=|\xi|^{-1}} \psi^{-1}\left(x\xi + x^{-1}\right) \mathrm{d}x & \text{for} \quad |\xi| > q. \end{cases}$$

To calculate its Fourier transform, observe that the Kloosterman integrals are related to the Fourier transform of $1_{\mathfrak{o}_{\mathrm{F}}}(x)\psi^{-1}\left(\dfrac{1}{x}\right)$:

Lemma 2.7.6.

$$\mathbb{F}_\psi\left(1_{\mathfrak{o}_{\mathrm{F}}}(x)\psi^{-1}\left(\frac{1}{x}\right)\right)(y) = \begin{cases} 1-q^{-1}-q^{-2} & \textit{for} \quad |y| \leq 1; \\ -q^{-1}-q^{-2} & \textit{for} \quad |y| = q; \\ \int_{|x|^2=|y|^{-1}} \psi^{-1}\left(\frac{1}{y} + xy\right) \mathrm{d}x & \textit{for} \quad |y| > q. \end{cases}$$

Proof. Since $1_{\mathfrak{o}_{\mathrm{F}}}(x)\psi^{-1}\left(\dfrac{1}{x}\right) \in \mathcal{L}^1(\mathrm{F}) \cap \mathcal{L}^2(\mathrm{F})$, we have

$$\begin{aligned} \mathbb{F}_\psi\left(1_{\mathfrak{o}_{\mathrm{F}}}(x)\psi^{-1}\left(\frac{1}{x}\right)\right)(y) &= \int_{\mathrm{F}} 1_{\mathfrak{o}_{\mathrm{F}}}(x)\psi^{-1}\left(\frac{1}{x}\right)\psi^{-1}(xy)\,\mathrm{d}x \\ &= \int_{\mathfrak{o}_{\mathrm{F}}} \psi^{-1}\left(\frac{1}{x} + xy\right) \mathrm{d}x. \end{aligned}$$

When $|y| \leq 1$, we have $\psi^{-1}(xy) = 1$ for all $x \in \mathfrak{o}_{\mathrm{F}}$, hence

$$\begin{aligned} \int_{\mathfrak{o}_{\mathrm{F}}} \psi^{-1}\left(\frac{1}{x} + xy\right) \mathrm{d}x = \int_{\mathfrak{o}_{\mathrm{F}}} \psi^{-1}\left(\frac{1}{x}\right) \mathrm{d}x &= \int_{|x|\geq 1} |x|^{-2}\psi^{-1}(x)\,\mathrm{d}x \\ &= \int_{|x|=1} \mathrm{d}x + q^{-2}\int_{|x|=q} \psi^{-1}(x)\,\mathrm{d}x \\ &= (1-q^{-1}) + q^{-2}\cdot(-1). \end{aligned}$$

When $|y| = q$, then $\psi^{-1}(xy) = 1$ for all $|x| \leq q^{-1}$, hence

$$\begin{aligned} \int_{\mathfrak{o}_{\mathrm{F}}} \psi^{-1}\left(\frac{1}{x} + xy\right) \mathrm{d}x &= \int_{|x|=1} \psi^{-1}(xy)\,\mathrm{d}x + \int_{|x|\leq q^{-1}} \psi^{-1}\left(\frac{1}{x}\right) \mathrm{d}x \\ &= -q^{-1} + \int_{|x|\geq q} \psi^{-1}(x)|x|^{-2}\,\mathrm{d}x = -q^{-1} - q^{-2}. \end{aligned}$$

When $|y| \geq q$, it follows from, for example, the last paragraph in [Sak13a, Section 5]. $\square$

Lemma 2.7.7.

$$\begin{aligned}
&\mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot| \left(f_{L(\mathrm{Std},s_1)L(\mathrm{Std},s_2)} \left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix} \right) \right)(x) \\
&= (1-q^{-2})^{-1}(1-q^{-n})^{-1} \begin{cases} \psi^{-1}\left(\frac{1}{x}\right) + 1 & \textit{for} \quad |x| \le q^{-1}; \\ |x|^{-n}(1+q^{-n}) & \textit{for} \quad |x| \ge 1. \end{cases}
\end{aligned}$$

Proof. From Lemma 2.7.6,

$$\begin{aligned}
&(1-q^{-2})(1-q^{n-1})(1-q^{-n}) \cdot \iota \circ |\cdot| \left(f_{L(\mathrm{Std},s_1)L(\mathrm{Std},s_2)} \left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix} \right) \right)(\xi) \\
&- (1-q^{n-1}) \mathbb{F}_{\psi} \left(1_{\mathfrak{o}_F}(\cdot) \psi^{-1}\left(\frac{1}{\cdot}\right) \right)(\xi) \\
&= \begin{cases} |\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2} & \text{for} \quad |\xi| \le 1; \\ -q^{-n-1} + q^{-1} + q^{-2} - q^{n-2} & \text{for} \quad |\xi| = q; \\ 0 & \text{for} \quad |\xi| > q. \end{cases}
\end{aligned}$$

Now, let us first compute the Fourier transform of

$$f_1(\xi) := \left(|\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2} \right) 1_{\mathfrak{o}_F}(\xi).$$

By definition, when $|x| \le 1$,

$$\begin{aligned}
&\mathbb{F}_{\psi^{-1}}(f_1)(x) = \int_F f_1(\xi)\psi(\xi x)\, \mathrm{d}\xi \\
&= \int_{|\xi|\le 1} \left(|\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2} \right) \mathrm{d}\xi \\
&= \sum_{|\xi|=q^{-i}, i=0}^{\infty} q^{-i(n-1)} q^{-i}(1-q^{-1})(1-q^{-2n}) + q^{-2} + q^{-1} - 1 - q^{n-2} \\
&= (1-q^{-1})(1+q^{-n}) + q^{-2} + q^{-1} - 1 - q^{n-2} \\
&= -q^{-n-1} + q^{-n} + q^{-2} - q^{n-2}.
\end{aligned}$$

When $|x| > 1$,

$$\begin{aligned}
&\mathbb{F}_{\psi^{-1}}(f_1)(x) = \int_F f_1(\xi)\psi(\xi x)\, \mathrm{d}\xi \\
&= \left\{ \int_{|\xi| \le |x|^{-1}} + \int_{|\xi| = |x|^{-1} q} \right\} \left(|\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2} \right) \psi(\xi x)\, \mathrm{d}\xi
\end{aligned}$$

Write $|x| = q^k$ with $k > 0$, then

$$\begin{aligned}
&\int_{|\xi|\le|x|^{-1}} \left(|\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2}\right) \psi(\xi x)\, \mathrm{d}\xi \\
&= (1-q^{-2n}) \sum_{|\xi|=q^{-i}, i=k}^{\infty} q^{-i(n-1)} q^{-i}(1-q^{-1}) + |x|^{-1}(q^{-2}+q^{-1}-1-q^{n-2}) \\
&= |x|^{-n}(1+q^{-n})(1-q^{-1}) + |x|^{-1}(q^{-2}+q^{-1}-1-q^{n-2}).
\end{aligned}$$

And

$$\begin{aligned}
&\int_{|\xi|=|x|^{-1}q} \left(|\xi|^{n-1}(1-q^{-2n)}) + q^{-2} + q^{-1} - 1 - q^{n-2}\right) \psi(\xi x)\, \mathrm{d}\xi \\
&= -|x|^{-1} \left((|x|^{-1}q)^{n-1}(1-q^{-2n}) + q^{-2} + q^{-1} - 1 - q^{n-2}\right),
\end{aligned}$$

hence

$$\begin{aligned}
\mathbb{F}_{\psi^{-1}}(f_1)(x) &= |x|^{-n}(q^{-n} - q^{-1} + 1 - q^{n-1}) \\
&= |x|^{-n}(1+q^{-n})(1-q^{n-1}).
\end{aligned}$$

To summarize,

$$\mathbb{F}_{\psi^{-1}}(f_1)(x) = \begin{cases} -q^{-n-1} + q^{-n} + q^{-2} - q^{n-2} & \text{for} \quad |x| \le 1; \\ |x|^{-n}(1+q^{-n})(1-q^{n-1}) & \text{for} \quad |x| > 1. \end{cases}$$

Next set

$$f_2(\xi) := \begin{cases} -q^{-n-1} + q^{-1} + q^{-2} - q^{n-2} & \text{for} \quad |\xi| = q; \\ 0 & \text{for} \quad |\xi| \ne q. \end{cases}$$

When $|x| \le q^{-1}$, we have

$$\begin{aligned}
\mathbb{F}_{\psi^{-1}}(f_2)(x) &= \int_{|\xi|=q} (-q^{-n-1} + q^{-1} + q^{-2} - q^{n-2}) \psi(x\xi)\, \mathrm{d}\xi \\
&= (q-1)(-q^{-n-1} + q^{-1} + q^{-2} - q^{n-2}) \\
&= q^{-n-1} - q^{-n} - q^{-2} + 1 + q^{n-2} - q^{n-1}.
\end{aligned}$$

When $|x| \ge 1$, $\mathbb{F}_{\psi^{-1}}(f_2)(x) \ne 0$ only when $|x| = 1$, in which case

$$\mathbb{F}_{\psi^{-1}}(f_2)(x) = q^{-n-1} - q^{-1} - q^{-2} + q^{n-2}.$$

Therefore,

$$\mathbb{F}_{\psi^{-1}}(f_2)(x) = \begin{cases} q^{-n-1} - q^{-n} - q^{-2} + 1 + q^{n-2} - q^{n-1} & \text{for} \quad |x| \leq q^{-1}; \\ q^{-n-1} - q^{-2} - q^{-1} + q^{n-2} & \text{for} \quad |x| = 1; \\ 0 & \text{for} \quad |x| > 1. \end{cases}$$

Combine them together, and we have

$$\mathbb{F}_{\psi^{-1}}(f_1 + f_2)(x) = \begin{cases} 1 - q^{n-1} & \text{for} \quad |x| \leq q^{-1}; \\ q^{-n}(1 - q^{n-1}) & \text{for} \quad |x| = 1; \\ |x|^{-n}(1 + q^{-n})(1 - q^{n-1}) & \text{for} \quad |x| > 1. \end{cases}$$

Therefore,

$$\begin{aligned} &\mathbb{F}_{\psi^{-1}} \circ \iota \circ |\cdot| \left(f_{L(\mathrm{Std}, \mathrm{s}_1)L(\mathrm{Std}, \mathrm{s}_2)} \left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix} \right) \right)(x) \\ &= (1 - q^{-2})^{-1}(1 - q^{-n})^{-1} \begin{cases} \psi^{-1}\left(\frac{1}{x}\right) + 1 & \text{for} \quad |x| \leq q^{-1}; \\ 1 + q^{-n} & \text{for} \quad |x| = 1; \\ |x|^{-n}(1 + q^{-n}) & \text{for} \quad |x| > 1. \end{cases} \end{aligned}$$

$\square$

Proposition 2.7.8.

$$\begin{aligned} &\widetilde{\mathbb{T}} \circ \mathcal{T} \left(f_{L(\mathrm{Std}, \mathrm{s}_1)L(\mathrm{Std}, \mathrm{s}_2)} \left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix} \right) \right)(x) \\ &= (1 - q^{-2})^{-1}(1 - q^{-n})^{-1} \begin{cases} |x|^{-\frac{1}{2}} \left(\psi^{-1}\left(\frac{2}{x}\right) + \psi\left(\frac{2}{x}\right) \right) & \textit{for} \quad |x| \leq q^{-1}; \\ |x|^{-n-\frac{1}{2}}(1 + q^{-n}) & \textit{for} \quad |x| \geq 1. \end{cases} \end{aligned}$$

Proof. This is according to Proposition 2.3.16 and Lemma 2.7.7. $\square$

Finally, let $\mathbb{L}_{\mathrm{X}}^{\circ}$ be the basic measure on $\mathrm{X} \times \mathrm{X} /\!/ \mathrm{SO}_{2n+1} \cong \mathbb{A}^1$, then the image of Φ_0 to $\mathcal{S}(\mathfrak{X})$ is

$$(1 + q^{-n})\mathbb{L}_0 = \frac{\zeta_{\mathrm{F}}(n)}{\zeta_{\mathrm{F}}(2n)} \mathbb{L}_{\mathrm{X}}^{\circ}.$$

Combine Proposition 2.7.5 and Proposition 2.7.8, we have

Theorem 2.7.9. *Let $\mathbb{L}_0 \in \mathcal{S}(\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n})$ be the push-forward of $1_{\mathrm{X}(\mathfrak{o}_\mathrm{F})}\,\mathrm{d}x \otimes 1_{\mathrm{X}(\mathfrak{o}_\mathrm{F})}\,\mathrm{d}x$, then we have*

$$\mathcal{T}^{-1}\left(\mathbb{L}_\mathrm{X}^\circ\right) = \frac{q^{\dim \mathrm{X}}}{\#\overline{\mathrm{X}}(\kappa_\mathrm{F})\zeta_\mathrm{F}(2)\zeta_\mathrm{F}(s_\mathrm{X})} f_{L_\mathrm{X}} = \frac{\zeta_\mathrm{F}(2n)}{\zeta_\mathrm{F}(2)\zeta_\mathrm{F}(n)^2} f_{L\left(\mathrm{Std},\mathrm{n}-\frac{1}{2}\right)L\left(\mathrm{Std},\frac{1}{2}\right)}.$$

2.7.2 Comparison between $\mathrm{A}\backslash\mathrm{PGL}_2/(\mathrm{N},\psi)$ and $(\mathrm{N},\psi)\backslash\mathrm{PGL}_2/(\mathrm{N},\psi)$

Let A be the maximal torus of PGL_2 consisting of diagonal matrices. For any $s \in \mathbb{C}$, write $\mathcal{L}_s$ for the line bundle on $\mathrm{A}\backslash\mathrm{PGL}_2$ induced by the unramified character

$$\mathrm{A}_1 \to \mathbb{C}^\times : \begin{pmatrix} a & \\ & 1 \end{pmatrix} \mapsto |a|^s.$$

Write $\mathcal{C}_c^\infty(\mathrm{A}\backslash\mathrm{PGL}_2, \mathcal{L}_s)$ be the Schwartz sections of $\mathcal{L}_s$, which can be trivialized as functions Ψ on PGL_2 satisfying

$$\Psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) = |a|^s \Psi(g)$$

with compact support module A. Take Ψ_s° to be the *basic function* such that its support is $\mathrm{A}_1 \cdot \mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})$ and satisfying

$$\Psi_s\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) = |a|^s,\ \forall \begin{pmatrix} a & \\ & 1 \end{pmatrix} \in \mathrm{A} \text{ and } k \in \mathrm{PGL}_2(\mathfrak{o}_\mathrm{F}).$$

Consider the unfolding map

$$\begin{aligned}\mathrm{U}_\psi : \mathcal{C}_c^\infty(\mathrm{A}_1\backslash\mathrm{PGL}_2, \mathcal{L}_s) &\to \mathcal{C}^\infty(\mathrm{N}, \psi\backslash\mathrm{PGL}_2) \\ \Psi &\mapsto \left(\mathrm{PGL}_2 \ni g \mapsto \int_\mathrm{F} \Psi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g\right)\psi^{-1}(n)\,\mathrm{d}n\right).\end{aligned}$$

Note that $\mathrm{A}\backslash\mathrm{PGL}_2$ can be identified with the line bundle $\mathcal{O}(2)$ over $\mathrm{B}\backslash\mathrm{PGL}_2 = \mathbb{P}^1$, and $\mathrm{N}\backslash\mathrm{PGL}_2$ is then identified with the open PGL_2-orbit consisting of non-zero sections in the total space of the dual bundle $\mathcal{O}(-2)$ in the usual way, or see [Sak13a]. For any $s \in \mathbb{C}$, let $\mathbb{L}_s$ be the following line bundle over $\mathcal{O}(-2)$:

$$\mathbb{L}_s^\psi := \mathfrak{L}_s \otimes \mathcal{L}_\psi.$$

Here the line bundle $\mathcal{L}_\psi$ is the Whittaker bundle over $\mathrm{N}\backslash\mathrm{PGL}_2$, and the sections of the other line bundle $\mathfrak{L}_s$, are the smooth functions over $\mathrm{N}\backslash\mathrm{PGL}_2$, and in any small neighborhood of the boundary $(p,0)\in\mathcal{O}(-2)$, where $p\in\mathbb{P}^1$ and 0 means the zero cotangent vector at p, the sections are of the form $|a|^{s+1}\Phi(q,a)$ for some smooth function Φ near $(p,0)$. Then according to [SV17, Section 9.5, (9.17)], the unfolding map U gives an isomorphism between the Schwartz sections

$$\mathcal{F}(\mathrm{A}_1\backslash\mathrm{PGL}_2,\mathcal{L}_s)\cong\mathcal{F}(\mathcal{O}(-2),\mathbb{L}_s^{\psi}).$$

Lemma 2.7.10. *Let Ψ_s° be the basic function of $\mathcal{C}_c^\infty(\mathrm{A}_1\backslash\mathrm{PGL}_2,\mathcal{L}_s)$, then*

$$\mathrm{U}_\psi(\Psi_s^\circ)=\Phi_{L\left(\mathrm{Std},\mathrm{s}+\frac{1}{2}\right)}$$

Proof. See [Sak13a, Lemma 5.5]. □

Note that $\mathcal{F}(\mathrm{A}\backslash\mathrm{PGL}_2)=\mathcal{F}(\mathrm{A}\backslash\mathrm{PGL}_2,\mathcal{L}_0)$, which consists of usual Schwartz functions on $\mathrm{A}\backslash\mathrm{PGL}_2$. While analogous results hold for the more general $\mathcal{L}_s$, we restrict our attention to the case $\mathcal{L}_0$ for the purposes of the present work.

Lemma 2.7.11. *Let $\Psi_1\in\mathcal{F}(\mathrm{A}\backslash\mathrm{PGL}_2)$ and $\Psi_2\in\mathcal{F}(\mathrm{N}\backslash\mathrm{PGL}_2,\mathbb{L}_s^{\psi^{-1}})$, when $\xi\in\mathrm{F}$, the orbital integral*

$$\mathcal{O}_\xi(\Psi_1\otimes\Psi_2):=\int_{\mathrm{PGL}_2}\Psi_1\left(\mathbf{w}\begin{pmatrix}1&\xi\\&1\end{pmatrix}g\right)\Psi_2(g)\,\mathrm{d}g$$

is absolutely convergent when $\mathrm{Re}(s)>-1$. Write $\mathcal{S}^-_{L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{A}\backslash\mathrm{PGL}_2/\mathrm{N},\psi)$ to be the space of measures on F of the form $\mathcal{O}_\xi(\Psi_1\otimes\Psi_2)\,\mathrm{d}\xi$ for $\Psi_1\in\mathcal{C}_c^\infty(\mathrm{A}\backslash\mathrm{PGL}_2)$ and $\Psi_2\in\mathcal{C}_c^\infty(\mathrm{N}\backslash\mathrm{PGL}_2,\mathbb{L}_s^{\psi^{-1}})$.

Remark 2.7.12. *The subscript L-value signifies the use of non-standard test measures derived from the Whittaker space. In the subsequent unfolding of $\mathcal{C}_c^\infty(\mathrm{A}\backslash\mathrm{PGL}_2)$, a second $L\left(\mathrm{st},\frac{1}{2}\right)$ is contributed. Specifically, the reader should view the space*

$$\mathcal{S}^-_{L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{A}\backslash\mathrm{PGL}_2/\mathrm{N},\psi)$$

in relation to $\mathcal{S}^-_{L\left(\mathrm{st},\frac{1}{2}\right)L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{PGL}_2/\mathrm{N},\psi)$, a comparison which constitues the primary objective of this section.

Proof. Using the Iwasawa decomposition, we have

$$\int_{\mathrm{PGL}_2} \Psi_2\left(\mathbf{w}\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix} g\right) \Psi_2(g)\,\mathrm{d}g$$

$$= \int_{\mathrm{F}\times\mathrm{F}^\times\times\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})} \Psi_1\left(\mathbf{w}\begin{pmatrix}1 & \xi+n \\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix} k\right) \psi^{-1}(n)\Psi_2\left(\begin{pmatrix}a & \\ & 1\end{pmatrix} k\right) |a|^{-1}\,\mathrm{d}n \frac{\mathrm{d}a}{|a|}\,\mathrm{d}k.$$

Note that

$$\Psi_1\left(\mathbf{w}\begin{pmatrix}1 & \xi+n \\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix} k\right) = \Psi_1\left(\mathbf{w}\begin{pmatrix}1 & \frac{\xi+n}{a} \\ & 1\end{pmatrix} k\right),$$

and since Ψ_1 is a Schwartz function, for fixed ξ, there is some $M > 0$ such that $\Psi_1 \neq 0$ only when $|\xi + n| < |a|M$. As Ψ_2 is also a Schwartz section, we have

$$\int_{\mathrm{PGL}_2} \Psi_2\left(\mathbf{w}\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix} g\right) \Psi_2(g)\,\mathrm{d}g \ll \int_{|a|\le 1} |a||a|^{s+1}|a|^{-1}\frac{\mathrm{d}a}{|a|} < \infty$$

when $\mathrm{Re}(s) > -1$.

$\square$

On the other hand, let $\Phi_1 \in \mathcal{F}(\mathcal{O}(-2), \mathbb{L}_0^{\psi})$ and $\Phi_2 \in \mathcal{F}(\mathcal{O}(-2), \mathbb{L}_s^{\psi^{-1}})$, for $\xi \in \mathrm{F}^\times$,

Lemma 2.7.13. *When* $\mathrm{Re}(s) > 0$*, the orbital integral*

$$\mathcal{O}_\xi(\Phi_1 \otimes \Phi_2) := \int_{\mathrm{PGL}_2} \Phi_1\left(\mathbf{w}\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right) \Phi_2(g)\,\mathrm{d}g,$$

is absolutely convergent. Moreover, the twisted-push forward measure

$$\mathcal{O}_\xi(\Phi_1 \otimes \Phi_2)\,\mathrm{d}\xi$$

lies in $\mathcal{S}^-_{L(\mathrm{st},s+\frac{1}{2})L(\mathrm{st},\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{PGL}_2/\mathrm{N},\psi)$.

Proof. According to the Iwasawa decomposition, we have

$$\int_{\mathrm{PGL}_2} \Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right) \Phi_2(g)\,\mathrm{d}g$$

$$= \int_{\mathrm{F}\times\mathrm{F}^\times\times\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})} \Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix}\xi & \\ & 1\end{pmatrix}\begin{pmatrix}1 & n \\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix} k\right)$$

$$\cdot\,\Phi_2\left(\begin{pmatrix}a & \\ & 1\end{pmatrix} k\right) \psi^{-1}(n)|a|^{-1}\,\mathrm{d}n\frac{\mathrm{d}a}{|a|}\,\mathrm{d}k.$$

As Φ_2 is a Schwartz section, there is some $M > 0$ such that $|a| < M$ and near 0 we have $\Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) \ll |a|^{s+1}$. Note that

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & a\xi \end{pmatrix}\begin{pmatrix} 1 & \\ -a^{-1}n & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

when $|n| \leq |a|$, we have

$$\Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) = \Phi_1\left(\begin{pmatrix} a^{-1}\xi^{-1} & \\ & 1 \end{pmatrix} k'\right)$$

for some $k' \in \mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})$, then since Φ_1 is a Schwartz section, there is some $N > 0$ such that $|a^{-1}\xi^{-1}| < N$ when $\Phi_1\left(\begin{pmatrix} a^{-1}\xi^{-1} & \\ & 1 \end{pmatrix} k'\right) \neq 0$, hence the integral is absolutely convergent in this part. Moreover, it is clear that when $\xi \leq \dfrac{1}{MN}$,

$$\int_{\mathrm{F}^\times \times \mathrm{F} \times \mathrm{PGL}_2(\mathfrak{o}_\mathrm{F}), |n| \leq |a|} \Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right)$$
$$\cdot \Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) \psi^{-1}(n)|a|^{-1}\, \mathrm{d}n \frac{\mathrm{d}a}{|a|}\, \mathrm{d}k = 0.$$

When $\xi \to \infty$, we may first assume Φ_1 and Φ_2 are $\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})$-invariant without loss of generality. Over the domain $\{(n, a) \in \mathrm{F} \times \mathrm{F}^\times \mid |n| \leq |a| \leq 1\}$, write

$$f_\xi(n, a) = \Phi_1\left(\begin{pmatrix} a^{-1}\xi^{-1} & \\ & 1 \end{pmatrix}\right) \Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{-1}$$

and

$$g_\xi(n, a) = |a|^{-1}\Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) \begin{cases} \frac{C}{|a\xi|} & \text{for} \quad |a| \geq |\xi|^{-1}; \\ 0 & \text{for} \quad |a| < |\xi|^{-1}, \end{cases}$$

where

$$C = \lim_{a \to 0} |a|^{-1}\Phi_1\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right).$$

Then we have the pointwise convergence

$$\lim_{|\xi| \to \infty} (f_\xi(n, a) - g_\xi(n, a)) = 0$$

and $|f_\xi(n,a)| \ll \max \Phi_1 \cdot \Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{-1}$, $|g_\xi(n,a)| \ll C \cdot \Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{-1}$, both of which are integrable when $\mathrm{Re}(s) > 0$, hence by dominant convergence theorem,

$$\lim_{\xi\to\infty}\left(\int_{\mathrm{F}^\times\times\mathrm{F},|a|\leq|a|\leq 1} f_\xi(n,a) - g_\xi(n,a)\right)\mathrm{d}n\frac{\mathrm{d}a}{|a|} = 0.$$

As it is clear that there are some constants C_1 and C_2 such that

$$\int_{\mathrm{F}^\times\times\mathrm{F},|a|\leq|a|\leq 1} g_\xi(n,a)\,\mathrm{d}n\frac{\mathrm{d}a}{|a|} = C_1|\xi|^{-1} + C_2|\xi|^{-s-1},$$

and the integral of $f_\xi(n,a)$ is a linear combination of complex powers of $|\xi|$, this yields the required asymptotic expansion near ∞.

When $|n| > |a|$, note that

$$\begin{pmatrix} a^{-1}\xi^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ -a^{-1}n & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\xi^{-1}n^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} \xi^{-1}n^{-1} & \\ & -a^{-1}n \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & -an^{-1} \end{pmatrix},$$

then

$$\begin{aligned}
&\Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right) \\
&= \psi\left(-\frac{1}{n\xi}\right)\Phi_1\left(\begin{pmatrix} -\frac{a}{\xi n^2} & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & -an^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} k\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathrm{PGL}_2}\Phi_1\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right)\Phi_2(g)\,\mathrm{d}g \\
&= \int_{\mathrm{F}\times\mathrm{F}^\times\times\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})|n|>|a|} \psi^{-1}\left(n + \frac{1}{n\xi}\right)\Phi_1\left(\begin{pmatrix} -\frac{a}{\xi n^2} & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & -an^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} k\right) \\
&\cdot \Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right)\mathrm{d}n\frac{\mathrm{d}a}{|a|^2}\,\mathrm{d}k.
\end{aligned}$$

For fixed ξ, when $\mathrm{Re}(s) > 0$,

$$\begin{aligned}
&\int_{\mathrm{F}\times\mathrm{F}^\times\times\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})|n|>|a|}\left|\Phi_1\left(\begin{pmatrix} -\frac{a}{\xi n^2} & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & -an^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} k\right)\Phi_2\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} k\right)\right|\mathrm{d}n\frac{\mathrm{d}a}{|a|^2}\,\mathrm{d}k \\
&\ll \int_{|a|\leq 1}\left\{\int_{a<|n|<|a|^{\frac{1}{2}}} 1\,\mathrm{d}n + \int_{|n|\geq|a|^{\frac{1}{2}}} |n|^{-2}\,\mathrm{d}n\right\}|a|^{s+2}|a|^{-1}\frac{\mathrm{d}a}{|a|} \\
&\ll \int_{|a|\leq 1}(1+|a|^{-1})|a|^{s+1}\frac{\mathrm{d}a}{|a|} < \infty.
\end{aligned}$$

when $\mathrm{Re}(s) > 0$.

When $|\xi| \to 0$, make a change of variable that $n = \frac{1}{u}$, then the integral over u is non-zero if and only if $|u|^2 = |\xi|$, in which case the integral, up to a constant, is

$$\int_{|u|^2=|\xi|} \psi^{-1}\left(\frac{u}{\xi} + u^{-1}\right) \mathrm{d}u \cdot |\xi|^{-1}.$$

When $\xi \to \infty$, the expected asymptotic behavior follows from a similar argument using the dominant convergence theorem as above.

$\square$

Before proceeding to the analysis of the transfer operator between

$$\mathcal{S}^-_{L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{A}\backslash\mathrm{PGL}_2/\mathrm{N},\psi) \text{ and } \mathcal{S}^-_{L\left(\mathrm{st},s+\frac{1}{2}\right)L\left(\mathrm{st},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{PGL}_2/\mathrm{N},\psi),$$

we have the following lemma, which is analogous to [Sak13a, Proposition 2.10].

Lemma 2.7.14. *Let $\Psi_1, \Psi_2 \in \mathcal{C}_c^\infty(\mathrm{F})$. Write*

$$\mathcal{O}^1_\xi(\Psi_1 \otimes \Psi_2) := \int_{\mathrm{F}^\times} |a\xi|^{-1}\Psi_1(a^{-1}\xi^{-1})\Psi_2(a)\frac{\mathrm{d}a}{|a|},$$

and

$$\mathcal{O}^2_\xi(\Psi_1, \Psi_2) := \int_{\mathrm{F}^\times} \Psi_1(\xi a)\Psi_2(a)\frac{\mathrm{d}a}{|a|}.$$

Write $\mathcal{S}(\mathbb{A}^2/(\mathbb{G}_m^\nabla, |\cdot|))$ for the space of measures on F of the form $\mathcal{O}^1_\xi(\Psi_1 \otimes \Psi_2)\,\mathrm{d}\xi$ and $\mathcal{S}(\mathbb{A}^2/\mathbb{G}_m^\Delta)$ for the space of measures on F of the form $\mathcal{O}^2_\xi(\Psi_1 \otimes \Psi_2)\,\mathrm{d}\xi$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}_c^\infty(\mathrm{F}) \otimes \mathcal{C}_c^\infty(\mathrm{F}) & \xrightarrow{\mathbb{F}_\psi \otimes \mathrm{Id}} & \mathcal{C}_c^\infty(\mathrm{F}) \otimes \mathcal{C}_c^\infty(\mathrm{F}) \\ \Big\downarrow \mathcal{O}^2_\cdot(\cdot) & & \Big\downarrow \mathcal{O}^1_\cdot(\cdot) \\ \mathcal{S}(\mathbb{A}^2/\mathbb{G}_m^\Delta) & \xrightarrow{\quad\mathcal{G}\quad} & \mathcal{S}(\mathbb{A}^2/(\mathbb{G}_m^\nabla, |\cdot|)) \end{array},$$

where

$$\mathcal{G} = |\cdot|^{-1} \circ \iota \circ \mathbb{F}_\psi$$

Proof. Let $h(\xi) \in \mathcal{C}_c(\mathrm{F})$ be a test function on F, then we have

$$\begin{aligned} &\int_{\mathrm{F}} \mathcal{O}^1_\xi(\mathbb{F}_\psi(\Psi_1) \otimes \Psi_2)\overline{h(\xi)}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}}\int_{\mathrm{F}^\times} |a\xi|^{-1}\mathbb{F}_\psi(\Psi_1)(a^{-1}\xi^{-1})\Psi_2(a)\overline{h(\xi)}\frac{\mathrm{d}a}{|a|}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}^\times}\int_{\mathrm{F}} |\xi|\mathbb{F}_\psi(\Psi_1)(\xi)\Psi_2(a)\overline{h(a^{-1}\xi^{-1})}\frac{\mathrm{d}a}{|a|}\frac{1}{|a\xi^2|}\,\mathrm{d}\xi. \end{aligned}$$

According to the Plancherel formula for the Fourier transform,

$$\int_{\mathrm{F}} \mathbb{F}_{\psi}(\Psi_1)(\xi)\overline{h(a^{-1}\xi^{-1})}|\xi|^{-1}\,\mathrm{d}\xi = \int_{\mathrm{F}} \Psi_1(\xi)\overline{\mathbb{F}_{\psi^{-1}}(h(a^{-1}(\cdot)^{-1})|\cdot|^{-1})(\xi)}\,\mathrm{d}\xi.$$

Note that

$$\begin{aligned}\mathbb{F}_{\psi^{-1}}(h(a^{-1}(\cdot)^{-1})|\cdot|^{-1})(\xi) &= \int_{\mathrm{F}} h(a^{-1}x^{-1})|x|^{-1}\psi(x\xi)\,\mathrm{d}x \\ &= \int_{\mathrm{F}} h(x^{-1})|x|^{-1}\psi\left(x\xi a^{-1}\right)\,\mathrm{d}x \\ &= \mathbb{F}_{\psi^{-1}} \circ |\cdot|^{-1} \circ \iota(h)(\xi a^{-1}),\end{aligned}$$

hence

$$\begin{aligned}&\int_{\mathrm{F}} \mathcal{O}^1_{\xi}(\mathbb{F}_{\psi}(\Psi_1)\otimes\Psi_2)\overline{h(\xi)}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}^\times}\int_{\mathrm{F}} \Psi_1(\xi)\Psi_2(a)\overline{\mathbb{F}_{\psi^{-1}}\circ|\cdot|^{-1}\circ\iota(h)(\xi a^{-1})}\frac{\mathrm{d}a}{|a|^2}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}^\times} \Psi_1(\xi a)\Psi_2(a)\overline{\mathbb{F}_{\psi^{-1}}\circ|\cdot|^{-1}\circ\iota(h)(\xi)}\frac{\mathrm{d}a}{|a|}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}} \mathcal{O}^2_{\xi}(\Psi_1\otimes\Psi_2)\overline{\mathbb{F}_{\psi^{-1}}\circ|\cdot|^{-1}\circ\iota(h)(\xi)}\,\mathrm{d}\xi \\ &= \int_{\mathrm{F}} |\cdot|^{-1}\circ\iota\circ\mathbb{F}_{\psi}(\mathcal{O}^2_{\cdot}(\Psi_1\otimes\Psi_2))(\xi)\overline{h(\xi)}\,\mathrm{d}\xi.\end{aligned}$$

$\square$

Proposition 2.7.15. *Let $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$, then we have a commutative diagram*

$$\begin{array}{ccc}\mathcal{C}^\infty_c(\mathrm{A}\backslash\mathrm{PGL}_2)\otimes\mathcal{C}^\infty_c(\mathcal{O}(-2),\mathbb{L}^{\psi^{-1}}_s) & \xrightarrow{\mathrm{U}_\psi\otimes\mathrm{Id}} & \mathcal{C}^\infty_c(\mathcal{O}(-2),\mathbb{L}^{\psi}_0)\otimes\mathcal{C}^\infty_c(\mathcal{O}(-2),\mathbb{L}^{\psi^{-1}}_s) \\ \downarrow & & \downarrow \\ \mathcal{S}_{L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{A}_1\backslash\mathrm{PGL}_2/(\mathrm{N},\psi)) & \xrightarrow{\overline{\mathrm{U}}} & \mathcal{S}^-_{L\left(\mathrm{st},\frac{1}{2}\right)L\left(\mathrm{st},s+\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{PGL}_2/\mathrm{N},\psi)\end{array},$$

with $\overline{\mathrm{U}}$ given by the following formula on $\mathbb{A}^1$:

$$|\cdot|^{-1}\circ\iota\circ\mathbb{F}_{\psi}.$$

Proof. Let $\Psi_1 \in \mathcal{C}^\infty_c(\mathrm{A}\backslash\mathrm{PGL}_2)$ and $\Psi_2 \in \mathcal{C}^\infty_c(\mathcal{O}(-2),\mathbb{L}^{\psi^{-1}}_s)$, for $a \in \mathrm{F}^\times$ and $g \in \mathrm{PGL}_2$, write

$$\widehat{\Psi_1}(a,g) := \int_{\mathrm{F}} \Psi_1\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g\right)\psi^{-1}(an)\,\mathrm{d}n$$

and $\Psi_2(a,g) := \Psi_2\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} g\right)$. Then for $\xi \in \mathrm{F}^\times$, we have

$$\begin{aligned}\mathcal{O}_\xi(\mathrm{U}_\psi(\Psi_1)\otimes\Psi_2) &= \int_{\mathrm{PGL}_2} \mathrm{U}_\psi(\Psi_1)\left(\mathbf{w}\begin{pmatrix}\xi & \\ & 1\end{pmatrix} g\right)\Psi_2(g)\,\mathrm{d}g \\ &= \int_{\mathrm{A}\backslash\mathrm{PGL}_2}\int_{\mathrm{A}} \mathrm{U}_\psi(\Psi_1)\left(\mathbf{w}\begin{pmatrix}a\xi & \\ & 1\end{pmatrix} g\right)\Psi_2\left(\begin{pmatrix}a & \\ & 1\end{pmatrix} g\right)\,\mathrm{d}g.\end{aligned}$$

Note that

$$\mathrm{U}_\psi(\Psi_1)\left(\mathbf{w}\begin{pmatrix}a\xi & \\ & 1\end{pmatrix} g\right) = |a\xi|^{-1}\widehat{\Psi_1}(a^{-1}\xi^{-1},g),$$

hence

$$\mathcal{O}_\xi(\mathrm{U}_\psi(\Psi_1),\Psi_2) = \int_{\mathrm{A}\backslash\mathrm{PGL}_2}\int_{\mathrm{A}} |a\xi|^{-1}\widehat{\Psi_1}(a^{-1}\xi^{-1},g)\Psi_2(a,g)\,\mathrm{d}g.$$

According to the above lemma,

$$\int_{\mathrm{A}} |a\xi|^{-1}\widehat{\Psi_1}(a^{-1}\xi^{-1},g)\Psi_2(g) = |\cdot|^{-1}\circ\iota\circ\mathbb{F}_\psi(f)(\xi,g)$$

where

$$f(\xi,g) = \int_{\mathrm{A}}\Psi_1\left(\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix}\mathbf{w}\begin{pmatrix}a & \\ & 1\end{pmatrix} g\right)\Psi_2(g)\,\mathrm{d}a,$$

and hence

$$\mathcal{O}_\xi(\mathrm{U}_\psi(\Psi_1),\Psi_2) = |\cdot|^{-1}\circ\iota\circ\mathbb{F}_\psi(\mathcal{O}_\cdot(\Psi_1\otimes\Psi_2)).$$

$\square$

2.7.3 Comparison between $\mathrm{A}\backslash\mathrm{PGL}_2/(\mathrm{N},\psi)$ and $(\mathrm{N},\psi)\backslash\mathrm{Mp}_2/(\mathrm{N},\psi)$

The results of this section are due to [Jac87]. We present them here for the reader's convenience and to clarify the notation used throughout the paper.

Let us first recall the isomorphism

$$\widetilde{\lambda} : \mathcal{H}(\mathrm{PGL}_2,\mathrm{PGL}_2(\mathfrak{o}_\mathrm{F})) \cong \mathcal{H}(\mathrm{Mp}_2,\mathrm{K}')$$

as fixed in [Jac87, Section 2]. Let $s \in \mathbb{C}$, and consider the genuine character of the abelian subgroup of Mp_2

$$\left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}, \epsilon\right) \mapsto \gamma(a,\psi)^{-1}|a|^s\epsilon,$$

and consider the induced representation $\widetilde{\mathrm{I}}(\chi(s))$ of Mp_2 by left shift on the space of functions Φ such that

$$\Phi\left(g\left(\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}, \epsilon\right)\right) = |a|^{-1}\gamma(a,\psi)^{-1}|a|^s\epsilon\Phi(g).$$

Then the isomorphism $\widetilde{\lambda}$ is such that for any $h \in \mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}}))$, the action of h on the unramified vector in $\mathrm{I}(\chi(s))$ and $\widetilde{\lambda}(h)$ on the unramified vector in $\widetilde{\mathrm{I}}(\chi(s))$ are the same. Moreover, for any $m \geq 0$, let h_m be the characteristic function on $\mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})\begin{pmatrix} \varpi^m & \\ & 1\end{pmatrix}\mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})$, then $\widetilde{\lambda}(h_m)(g) = |a|^{\frac{1}{2}}\gamma(a,\psi)\epsilon$ if

$$g = \left(k_1\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}k_2, \epsilon\right)$$

with the valuation of a is m; $\widetilde{\lambda}(h_m)(g) = 0$ is g is not of this form.

Let $\mathcal{F}(\mathrm{N},\psi\backslash\mathrm{Mp}_2/\mathrm{N},\psi)$ be the space of functions on $\mathrm{F}^\times$ of the form

$$a \mapsto \int_{\mathrm{F}\times\mathrm{F}} \Phi\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\mathbf{w}\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & y \\ & 1\end{pmatrix}\right)\psi^{-1}(x+y)\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}_\xi(\Phi)$$

where Φ is a smooth function of compact support on Mp_2. Similarly let

$$\mathcal{F}(\mathrm{A}\backslash\mathrm{PGL}_2/\mathrm{N},\psi)$$

be the space of orbital integrals

$$\mathrm{F} \ni y \mapsto \int_{\mathrm{F}^\times\times\mathrm{F}} \Phi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & y \\ & 1\end{pmatrix}\mathbf{w}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right)\psi^{-1}(x)\frac{\mathrm{d}a}{|a|}\,\mathrm{d}x.$$

Proposition 2.7.16. *For any $h \in \mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}}))$, we have*

$$\mathcal{O}_a(\widetilde{\lambda}(h)) = |a|^{-1/2}\gamma(a,\psi)\psi\left(\frac{2}{a}\right)\mathcal{O}_{\frac{a}{4}}(h) =: \mathbb{J}(\mathcal{O}.(h))(a)$$

Proof. See [Jac87, Section 8]. ∎

Remark 2.7.17. *In terms of measures, we should multiply by* $|a|^{\frac{1}{2}}$ *instead of* $|a|^{-\frac{1}{2}}$*. And as a result, the diagram 2.11 is a commutative diagram.*

2.7.4 Comparison between $\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}/SO_{2n}$ and $(\mathrm{N},\psi)\backslash\mathrm{Mp}_2/(\mathrm{N},\psi)$

Lemma 2.7.18. *Let*

$$\mathrm{X}^\circ = \{v \in \mathrm{X} \mid \mathbb{B}_{\mathbb{V}}(v, e_1) \neq 0\}.$$

Then $v_0 \in \mathrm{X}^\circ$ *and* X° *is an open* B*-orbit in* X*.*

Proof. Same as Lemma 2.4.4. ∎

Corollary 2.7.19. $\mathrm{P}(\mathrm{X})$ *is the maximal proper parabolic subgroup fixing the isotropic flag*

$$\langle e_1 \rangle.$$

Then under the above basis $\{e_1, \cdots, e_n, f_n, \cdots, f_1\}$*, the Levi* $\mathrm{L}(\mathrm{X})$ *of* $\mathrm{P}(\mathrm{X})$ *is*

$$\mathrm{L}(\mathrm{X}) = \left\{ \begin{pmatrix} t & & \\ & g & \\ & & t^{-1} \end{pmatrix} \mid t \in \mathbb{G}_m, g \in \mathrm{SO}(v_0^\perp) \right\} \cong \mathbb{G}_m \times \mathrm{SO}(v_0^\perp),$$

and hence

$$\delta_{(\mathrm{X})}^{\frac{1}{2}} \left(\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \right) = |t_2|^{n-\frac{3}{2}} |t_3|^{n-\frac{5}{2}} \cdots |t_n|^{\frac{1}{2}}.$$

Moreover, the quotient map $\mathrm{A} \to \mathrm{A}_{\mathrm{X}} \cong \mathbb{G}_m$ *is given by*

$$\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto t_1.$$

Then after the Satake transform, $\mathbb{C}[\mathrm{A}^\vee]^{\mathbb{W}_\mathrm{G}} \to \mathbb{C}[\mathrm{A}_\mathrm{X}^\vee]^{\mathbb{W}_\mathrm{X}}$ *is given by*

$$\mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathbb{W}_{\mathrm{SO}_{2n}}} \to \mathbb{C}[x^{\pm}]^{\mathbb{Z}/2\mathbb{Z}} : x_1 \mapsto x,\ x_j \mapsto q^{j-n-\frac{1}{2}} \text{ for } 2 \leq j \leq n. \tag{2.12}$$

Now consider the nilpotent cone $\Sigma := \{v \in \mathbb{V} \mid \mathbb{B}_\mathbb{V}(v, v) = 0\}$.

Lemma 2.7.20.

$$\Sigma^\circ := \left\{ g \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid g \in \mathrm{SO}(\mathbb{V}) \right\}$$

is an open dense $\mathrm{SO}(\mathbb{V})$*-orbit of* Σ.

Proof. Same as Lemma 2.4.6. □

Lemma 2.7.21. *Let* $\Phi \in \mathcal{F}(\mathbb{Y})^{\mathrm{K}'}$*, and* $\omega_\psi(h)\Phi|_\Sigma = 0$ *for all* $h \in \mathrm{Mp}_2$*, then* $\Phi = 0$.

Proof. The same argument as in Lemma 2.4.7, noting that cuspidal genuine representations of Mp_2 are not spherical according to [FK86, Proposition 17]. □

Now let $\sigma \in \mathbb{C}$ be such that $\mathrm{Re}(\sigma) > 1$. Consider the following intertwining operators:

$$\mathrm{Z}_\sigma : \mathcal{F}(\mathbb{Y}) \to \mathcal{C}^\infty(\mathrm{SO}_{2n} \times \mathrm{Mp}_2) : \Phi \mapsto (g, h) \mapsto \int_\mathrm{F} \omega_\psi(g, h)^{-1} \Phi \left(\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) |a|^\sigma \frac{\mathrm{d}a}{|a|}.$$

which is absolutely convergent.

Proposition 2.7.22. *When* $\mathrm{Re}(\sigma) > 1$*,*

(1) Let $\mathfrak{s} = (s_1, s_2, \cdots, s_n) \in X^*(\mathrm{A}) \otimes_\mathbb{Z} \mathbb{C} = \mathbb{C}^n$*, write*

$$\chi(\mathfrak{s}) : \mathrm{T} \to \mathbb{C} : \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto \prod_{i=1}^n |t_i|^{s_i}$$

and $\mathrm{I}(\chi(\mathfrak{s}))$ *the correponding normalized induced representation of* SO_{2n}*. Then* Z_σ *factors through* $\mathrm{I}(\chi(n-\frac{1}{2}-\sigma, n-\frac{3}{2}, n-\frac{5}{2}, \cdots, \frac{1}{2}))\otimes\widetilde{\mathrm{I}}(\chi(\sigma-n+\frac{1}{2}))$ *as representations of* $\mathrm{SO}_{2n}\times\mathrm{Mp}_2$*.*

(2) *Write* K *for the maximal compact subgroup of* $\mathrm{SO}(\mathbb{V})$ *fixing the standard lattice* $\mathfrak{o}_\mathrm{F}^{2n}$ *under the ordered basis* $\{e_1, \cdots, e_n, w, f_n, \cdots, f_1\}$*. Let*

$$\lambda_\mathrm{X} : \mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K}) \to \mathcal{H}(\mathrm{Mp}_2, \mathrm{K}')$$

be the morphism of the Hecke algebra corresponding to the above. Let Φ_0 *be the characteristic function of* $\mathbb{V}(\mathfrak{o}_\mathrm{F})$*, then we have*

$$\omega_\psi(f^\vee)\Phi_0 = \omega_\psi(\lambda_\mathrm{X}(f))\Phi_0.$$

for all $f\in\mathcal{H}(\mathrm{SO}(\mathbb{V}), \mathrm{K})$*.*

Proof. (1) is simply a matter of computing the effect of $\mathrm{A}\times\mathrm{T}$ on Z_σ, noting that

$$\gamma(t, \psi_{\frac{1}{2}})^{-(2n+1)}(t, (-1)^n 2) = \gamma(t, \psi)^{-1}.$$

(2) is totally the same as in the previous sections. ☐

2.7.5 Comparison between $(\mathrm{N}, \psi)\backslash\mathrm{PGL}_2/(\mathrm{N}, \psi)$ and $\mathrm{SO}_{2n}\backslash\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$

Lemma 2.7.23. *Let* h_{Sym^m} *be the Satake inverse of the symmetric* m*-th power in the spherical Hecke algebra* $\mathcal{H}(\mathrm{PGL}_2, \mathrm{PGL}_2(\mathfrak{o}_\mathrm{F}))$*, then*

$$\Omega(\Phi_0)(g) = \frac{1}{\zeta_\mathrm{F}(n)}\sum_{m=0}^{\infty} q^{-m(n-\frac{1}{2})}\int_\mathrm{N}\widetilde{\lambda}(h_{\mathrm{Sym}^m})(ng)\psi^{-1}(n)\,\mathrm{d}n.$$

Proof. A first observation is that

$$\begin{aligned}\sum_{m=0}^{\infty} q^{-m(n-\frac{1}{2})}h_{\mathrm{Sym}^m} &= \sum_{m=0}^{\infty} q^{-mn}(h_m + h_{m-2}+\cdots) \\ &= \sum_{m=0}^{\infty}\frac{q^{-mn}}{1-q^{-2n}}h_m.\end{aligned}$$

According to [Jac87, Lemma on Page 143], we have

$$\int_\mathrm{N}\widetilde{\lambda}(h_m)\left(n\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}\right)\psi^{-1}(n)\,\mathrm{d}n = |a|^{\frac{1}{2}}\gamma(a,\psi)1_{\varpi^m\mathfrak{o}_\mathrm{F}^\times\cup\varpi^{m-1}\mathfrak{o}_\mathrm{F}^\times}(a),$$

we have when $|a| \leq 1$,

$$\sum_{m=0}^{\infty} \frac{q^{-mn}}{1-q^{-2n}} \int_{\mathrm{N}} \widetilde{\lambda}(h_m) \left(n \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \psi^{-1}(n)\, \mathrm{d}n$$
$$= \sum_{m=0}^{\infty} \frac{|a|^{\frac{2n+1}{2}}}{1-q^{-2n}} (1+q^{-n}) \gamma(a,\psi),$$

and 0 otherwise.

Also note that

$$\omega_{\psi} \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right) \Phi_0(v_0) = \begin{cases} \gamma(a,\psi)|a|^{\frac{2n+1}{2}} & \text{if } |a| \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then the Lemma follows. $\square$

Remark 2.7.24. *Formally,*

$$\Omega(\Phi_0) = \frac{1}{\zeta_{\mathrm{F}}(n)} \int_{\mathrm{N}}^{*} \widetilde{\lambda} \left(\Phi_{L\left(\mathrm{st}, n-\frac{1}{2}\right)} \right) (ng) \psi^{-1}(n)\, \mathrm{d}n.$$

Theorem 2.7.25. *Conjecture 2.1.5 is true for* X *of type* B_n.

Proof. Let us start with the basic function $1_{(\mathrm{N},\psi)\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \otimes \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)L\left(\mathrm{st},\frac{1}{2}\right)}$, and for any $h \in \mathcal{H}(\mathrm{SO}(\mathbb{V}),\mathrm{K})$, we have

$$\mathcal{O}_{\xi} \left(1_{(\mathrm{N},\psi)\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \lambda_{\mathrm{X}}(h), \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)L\left(\mathrm{st},\frac{1}{2}\right)} \right) = \mathcal{O}_{\xi} \left(\Phi_{L\left(\mathrm{st},\frac{1}{2}\right)} \star \lambda_{\mathrm{X}}(h), \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)} \right)$$

according to the commutativity of the Hecke algebra. Then, according to the unfolding,

$$\mathcal{O}_{\xi} \left(\Phi_{L\left(\mathrm{st},\frac{1}{2}\right)} \star \lambda_{\mathrm{X}}(h), \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)} \right) = \overline{\mathrm{U}}^{-1} \left(\mathcal{O}_{\xi} \left(1_{\mathrm{A}\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \lambda_{\mathrm{X}}(h), \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)} \right) \right).$$

According to the above lemma,

$$\mathcal{O}_{\xi} \left(1_{\mathrm{A}\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \lambda_{\mathrm{X}}(h), \Phi_{L\left(\mathrm{st},n-\frac{1}{2}\right)} \right)$$
$$= \sum_{m=0}^{\infty} q^{-mn} \mathcal{O}_{\xi} \left(1_{\mathrm{A}\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \mathrm{Sat}^{-1}(\mathrm{Sym}^m) \star \lambda_{\mathrm{X}}(h), 1_{(\mathrm{N},\psi)\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \right).$$

Then by Proposition 2.7.16,

$$\mathcal{O}_{\xi} \left(1_{\mathrm{A}\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \mathrm{Sat}^{-1}(\mathrm{Sym}^m) \star \lambda_{\mathrm{X}}(h), 1_{(\mathrm{N},\psi)\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \right)$$
$$= \mathbb{J}^{-1} \left(\mathcal{O}_{\xi} \left(\widetilde{\lambda}(\mathrm{Sat}^{-1}(\mathrm{Sym}^m) \star \lambda_{\mathrm{X}}(h)) \right) \right).$$

Since the orbital integral is absolutely convergent, combined with Lemma 2.7.23,

$$\sum_{m=0}^{\infty} q^{-mn} \mathbb{J}\left(\mathcal{O}_{\xi}\left(1_{\mathrm{A}\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})} \star \mathrm{Sat}^{-1}(\mathrm{Sym}^m) \star \lambda_{\mathrm{X}}(h), 1_{(\mathrm{N},\psi)\backslash \mathrm{PGL}_2(\mathfrak{o}_{\mathrm{F}})}\right)\right)$$
$$= \zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(n) \int_{\mathrm{N}}^{*} \omega_{\psi}\left(\mathbf{w}\begin{pmatrix}\zeta & \\ & 1\end{pmatrix} n\right) \omega_{\psi}(\widetilde{\lambda}(\lambda_{\mathrm{X}}(h)))\Phi_0(v_0)\psi^{-1}(n)\,\mathrm{d}n,$$

which is

$$\frac{\zeta_{\mathrm{F}}(2)\zeta_{\mathrm{F}}(n)^2}{\zeta_{\mathrm{F}}(2n)} \mathcal{T}^{-1}(\mathbb{L}_{\mathrm{X}}^{\circ} \star h)$$

due to Proposition 2.7.22.

□

Chapter 3

Braverman-Kazhdan-Ngô Proposal

3.1 Introduction

Let G be a split reductive algebraic group defined over a number field k and $\mathrm{G}^\vee$ be the Langlands dual group of G. For an irreducible finite-dimensional representation (ρ, V_ρ) of $\mathrm{G}^\vee(\mathbb{C})$ and an irreducible cuspidal automorphic representation π of $\mathrm{G}(\mathbb{A})$, R. Langlands introduced in [Lan70] the automorphic L-function $L(s,\pi,\rho)$ associated with (π,ρ) and proved that $L(s,\pi,\rho)$ can be defined as an Euler product of local L-factors

$$L(s,\pi,\rho) = \prod_{\nu\in|k|} L_\nu(s,\pi_\nu,\rho)$$

when $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently positive, where $|k|$ is the set of all local places of k, $\pi = \otimes'_\nu \pi_\nu$ (the restricted tensor product decomposition), and the local L-factors $L_\nu(s,\pi_\nu,\rho)$ can be defined through the local Langlands conjecture. It is well known that the local L-factors are well-defined when ν is archimedean or any finite local place at which π_ν is unramified. The Langlands conjecture predicts that $L(s,\pi,\rho)$ admits a meromorphic continuation to $s \in \mathbb{C}$ and satisfies the standard global functional equation. One of the central problems in the theory of automorphic forms and in the Langlands program is to prove the Langlands conjecture for general automorphic L-functions $L(s,\pi,\rho)$.

The Langlands-Shahidi method and the Rankin-Selberg method have established the Langlands conjecture for numerous cases. However, the general situation of the Langlands conjecture remains largely open. Around the year 2000, A. Braverman and D. Kazhdan ([BK00]) proposed a framework to establish the Langlands conjecture for general L-functions $L(s,\pi,\rho)$ via local and global harmonic analysis on G, which generalizes the method of J. Tate's thesis and the work of R. Godement and H. Jacquet to the great generality. In the Braverman-Kazhdan proposal [BK00] and in its refinement by B. C. Ngô in [Ngo20], the pair (G,ρ) should satisfy the following properties.

Definition 3.1.1 (Braverman–Kazhdan–Ngo Pair). *Let G be a split reductive group over a field of characteristic zero and let $\rho : \mathrm{G}^\vee \to \mathrm{GL}_{V_\rho}$ be a representation of its dual group. The pair (G,ρ) is a BKN pair if it satisfies the following conditions.*

1. *The kernel of $\rho\colon \mathrm{G}^\vee \to \mathrm{GL}_{d_\rho}$ is trivial, where d_ρ is the dimension of ρ.*
2. *There exists a character $\mathfrak{d}\colon \mathrm{G} \to \mathbb{G}_m$, such that the following exact sequence*
$$1 \longrightarrow \mathrm{G}_0 \longrightarrow \mathrm{G} \xrightarrow{\mathfrak{d}} \mathbb{G}_m \longrightarrow 1$$
holds with $\mathrm{G}_0 = [\mathrm{G},\mathrm{G}]$ the derived group of G.
3. *The derived group $\mathrm{G}_0 = [\mathrm{G},\mathrm{G}]$ is simply connected.*
4. *The dual morphism $\mathfrak{d}^\vee\colon \mathbb{G}_m \to \mathrm{G}^\vee$ composed with the given representation ρ is a scalar multiplication of $\mathbb{G}_m$ on the vector space V_ρ, i.e., the identity $\rho(\mathfrak{d}^\vee(a)) = a \cdot \mathrm{I}_{V_\rho}$ holds for any $a \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$.*

Note that in Definition 3.1.1, we take any field of characteristic zero as the ground field, which puts the notion in algebraic terms, and keep in mind that the notion of the BKN pairs may have a connection to the Relative Langland Duality of D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh ([BZSV24]).

The local conjecture in the Braverman-Kazhdan proposal for any local field F of characteristic zero can be found in [Ngo20, Section 4] and [LN24, Section 2.1], which we will also review in Section 3.2. The assumptions for a BKN pair are natural but restrictive. For the Langlands conjecture on automorphic L-function, one may want to consider the following general pairs (G,ρ).

Definition 3.1.2 (L-pair)**.** *Let* G *be a reductive group that splits over a number field* k *and let* $\rho : \mathrm{G}^\vee \to \mathrm{GL}_{V_\rho}$ *be a representation of its dual group. We say* (G, ρ) *is an* L*-pair if the restriction of* ρ *to any essentially simple component of the derived group of* $\mathrm{G}^\vee$ *is nontrivial.*

In the context of the theory of L-functions, it is sufficient to restrict our attention to the Braverman–Kazhdan–Ngo pairs due to the following Theorem in [JLX26].

Theorem 3.1.3. *Let* F *be a local field of characteristic zero, let* G *be a split reductive* F*-group, and let* (G, ρ) *be an* L*-pair. Then there is a canonical BKN pair* $(\widetilde{\mathrm{G}}, \widetilde{\rho})$ *attached to* (G, ρ) *with the following property: Assume that the local Langlands correspondence is known for* $\mathrm{G}(F)$ *and* $\widetilde{\mathrm{G}}(F)$*. If* π *is an irreducible admissible representation of* $\mathrm{G}(F)$*, then there is an irreducible admissible representation* σ *of* $\widetilde{\mathrm{G}}(F)$ *such that the local* L*-factors and local* ϵ*-factors:*

$$L(s, \pi, \rho) = L(s, \sigma, \widetilde{\rho}) \quad \text{and} \quad \epsilon(s, \pi, \rho, \psi) = \epsilon(s, \sigma, \widetilde{\rho}, \psi)$$

hold as meromorphic functions of s *for all non-trivial characters* $\psi : F \to \mathbb{C}^\times$*.*

Example 3.1.4. *Let* $\mathrm{G} = \mathrm{GL}_n$*,* $\rho = \mathrm{Std}$ *the standard representation of the dual group* $\mathrm{G}^\vee = \mathrm{GL}_n$*, and* $\mathfrak{d} = \det : \mathrm{GL}_n \to \mathbb{G}_m$*, then one can check easily that* $(\mathrm{GL}_n, \mathrm{Std})$ *is a Braverman-Kazhdan-Ngo pair.*

For more examples, we refer the reader to the recent work in [JLX26].

3.2 Local Theory: Non-standard Test Measures and Fourier Transforms

Note that although the Braverman-Kazhdan proposal can be formulated for general reductive groups defined over a number filed k, for the purpose of this thesis, we assume that G is k-split.

The local conjecture in the Braverman-Kazhdan proposal for any local field F of characteristic zero can be stated as follows. Some more detailed comments on this conjecture can be found in [Ngo20, Section 4] and [LN24, Section 2.1].

Conjecture 3.2.1 (Braverman-Kazhdan)**.** *Let F be a local field of characteristic zero and (G, ρ) be a Braverman–Kazhdan–Ngo pair. Let π be an irreducible admissible representation of $\mathrm{G}(\mathrm{F})$, which is of the Casselman-Wallach type if F is archimedean. Then the following statements hold.*

1. **Local Zeta Integral:** *There exists a Schwartz space $\mathcal{S}_\rho(\mathrm{G}(F))$ with*

$$\mathcal{C}_c^\infty(\mathrm{G}(\mathrm{F})) \subset \mathcal{S}_\rho(\mathrm{G}(\mathrm{F})) \subset \mathcal{C}^\infty(\mathrm{G}(\mathrm{F}))$$

such that the local zeta integral

$$\mathcal{Z}(s, \varphi_\pi, \phi) = \int_{G(F)} \varphi_\pi(g)\phi(g)|\mathfrak{d}(g)|_F^{s+\frac{n_\rho - 1}{2}}\,\mathrm{d}g, \quad \text{for } \varphi_\pi \in \mathcal{C}(\pi) \text{ and } \phi \in \mathcal{S}_\rho(\mathrm{G}(\mathrm{F}))$$

converges absolutely for $\mathrm{Re}(s)$ sufficiently positive and admits a meromorphic continuation to $s \in \mathbb{C}$, where n_ρ is the normalization as given in [Ngo20], $\mathcal{C}(\pi)$ is the space of matrix coefficients of π, $\mathcal{C}^\infty(\mathrm{G}(\mathrm{F}))$ is the space of smooth functions on $\mathrm{G}(\mathrm{F})$ and $\mathcal{C}_c^\infty(\mathrm{G}(\mathrm{F}))$ is its subspace of compactly supported functions.

2. **Local L-Factor:** *The local zeta integrals $\mathcal{Z}(s, \varphi_\pi, \phi)$ are holomorphic multiples of the local Langlands L-factor $L(s, \pi, \rho)$. In the p-adic case, the space*

$$\mathcal{I}(\pi) = \{\mathcal{Z}(s, \varphi_\pi, \phi) \mid \varphi_\pi \in \mathcal{C}(\pi) \text{ and } \phi \in \mathcal{S}_\rho(\mathrm{G}(F))\}$$

is a non-zero fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ generated by $L(s, \pi, \rho)$, where q is the cadinarity of the residue field of F. In the Archimedean case, for any vertical strip

$$S_{a,b} = \{s \in \mathbb{C} \mid a < \mathrm{Re}(s) < b\}$$

with $a < b$, if $p(s) \in \mathbb{C}[s]$ is such polynomials that $p(s)L(s, \pi, \rho)$ is bounded on $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s, \pi, \rho)$ removed, then $p(s)\mathcal{Z}(s, \varphi_\pi, \phi)$ are also bounded on $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s, \pi, \rho)$ removed, for any $\varphi_\pi \in \mathcal{C}(\pi)$ and $\phi \in \mathcal{S}_\rho(\mathrm{G}(\mathrm{F}))$.

3. **Basic Function:** *There is a basic function $\mathbb{L}_\rho \in \mathcal{S}_\rho(\mathrm{G}(\mathrm{F}))$ such that when π is unramified with a normalized zonal spherical function $\varphi_\pi^\circ \in \mathcal{C}(\pi)$, the identity*

$$\mathcal{Z}(s, \varphi_\pi^\circ, \mathbb{L}_\rho) = L(s, \pi, \rho).$$

4. **Fourier Transform:** *There exists an invariant distribution* J^{ρ}_{ψ} *on* $\mathrm{G}(F)$ *with a non-trivial additive character* ψ *of* F *such that the convolution operator*

$$\mathcal{F}_{\rho,\psi}(\phi)(g) = |\mathfrak{d}(g)|^{-n_\rho}(\mathrm{J}^{\rho}_{\psi} * \phi^{\vee})(g), \quad \text{with } \phi \in \mathcal{S}_\rho(\mathrm{G}(\mathrm{F})) \text{ and } g \in \mathrm{G}(\mathrm{F})$$

defines a Fourier transform on $\mathcal{S}_\rho(\mathrm{G}(\mathrm{F}))$*, which enjoys the following properties:*

$$\mathcal{F}_{\rho,\phi}(\mathbb{L}_\rho) = \mathbb{L}_\rho,\ \mathcal{F}_{\rho,\psi} \circ \mathcal{F}_{\rho,\psi^{-1}} = \mathrm{I}.$$

Here I *is the identity operator. And* $\mathcal{F}_{\rho,\psi}$ *extends to a unitary operator on*

$$\mathcal{L}^2(\mathrm{G}(\mathrm{F}),\, \mathrm{d}_\rho g),$$

where $\mathrm{d}_\rho(g) = |\mathfrak{d}(g)|^{n_\rho}\,\mathrm{d}g$ *and* $\phi^{\vee}(g) = \phi(g^{-1})$.

5. **Local γ-Factor:** *The local functional equation*

$$\mathcal{Z}(1-s, \varphi_\pi^{\vee}, \mathcal{F}_{\rho,\psi}(\phi)) = \gamma(s,\pi,\rho,\psi)\mathcal{Z}(s,\varphi_\pi,\phi), \quad \text{for } \varphi_\pi \in \mathcal{C}(\pi) \text{ and } \phi \in \mathcal{S}_\rho(\mathrm{G}(\mathrm{F})),$$

holds with the Langlands local γ*-factor* $\gamma(s,\pi,\rho,\psi)$*, as meromorphic functions in* $s \in \mathbb{C}$.

Remark 3.2.2. *Depending on the chosen convention, some authors formulate the* ρ*-Fourier transform as a map from* $\mathcal{S}_\rho(\mathrm{G}(\mathrm{F}))$ *to* $\mathcal{S}_{\rho^\vee}(\mathrm{G}(\mathrm{F}))$*, for example in [Sak23a]. The two formulations are essentially equivalent.*

Remark 3.2.3. *The* ρ*-Schwartz function is expected to be connected with a reductive monoid* M^ρ *containing* G *as the open subset of invertible elements. The theory of reducitve monoid over an algebraically closed field is developed in [Put88, Vin95, Ren05], and an explicit construction over a non-algebraically closed field is given in [Ngo20]. For a comparison of the theories, see also [JLX26]. Over a non-Archimedean local field of positive characteristic, the basic function* $\mathbb{L}^\rho$ *was constructed in [BNS16]. In general, it is expected that the* ρ*-Schwartz space can be constructed by nearby cycles in [BBFK24].*

Example 3.2.4. *In the case where* G *is a torus, Conjecture 3.2.1 has been established for arbitrary* ρ*. This result was first outlined in [Ngo20] and subsequently developed in greater detail in [LN24].*

Example 3.2.5. *In the case* $\mathrm{G} = \mathrm{GL}_n$ *and* $\rho = \mathrm{Std}$*, Conjecture 3.2.1 was established first by J. Tate in his famous thesis [Tat67] for the case* $n = 1$ *and was generalized by R. Godement and H. Jacquet for general* n *in [GJ72, Jac09]. In this case,* $\mathcal{S}_{\mathrm{Std}}(\mathrm{GL}_n(\mathrm{F}))$ *consists of all locally constant functions with compact support on* $\mathrm{Mat}_n(\mathrm{F})$*, the* $n \times n$ *matrices. The* ρ*-Fourier transform is the standard Fourier transform with* $\mathrm{J}_{\psi}^{\mathrm{Std}}(g) = \psi(\mathrm{tr}(g))$*.*

Example 3.2.6. *In the case that* $\mathrm{G} = \mathrm{Sp}_{2n} \times \mathbb{G}_m$ *and* $\rho = \mathrm{Std}$*, the Conjecture 3.2.1 is known over non-Archimedean local field based on [JLZ24].*

Example 3.2.7. *The explicit construction of the Fourier kernel for the symmetric power of* GL_2 *was recently established in [LN24], which is particularly significant as it provides a family of examples involving higher-dimensional representations.*

Motivated by the global application of the trace formula, descending to test measures for the Arthur-Selberg or the Kuznetsov trace formulae, the ρ-Fourier transform would give rise to Hankel transforms

$$\mathcal{H}_\rho : \mathcal{S}_\rho\left(\frac{\mathrm{G}}{\mathrm{G}}\right) \to \mathcal{S}_{\rho^\vee}\left(\frac{\mathrm{G}}{\mathrm{G}}\right)$$

or

$$\mathcal{H}_\rho : \mathcal{S}_\rho(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi) \to \mathcal{S}_{\rho^\vee}(\mathrm{N}, \psi\backslash\mathrm{G}/\mathrm{N}, \psi)$$

between non-standard test measures for the corresponding trace formula. Ideally, if we could further prove a Poisson summation formula on the level of orbital integrals, responsible for the Hankel transforms, that would give rise to functional equations of the L-functions corresponding to ρ.

Remark 3.2.8. *For the purpose of Hankel transforms, following [Sak19] it is easier to work with half-densities, but one may also feel free to transfer densities with functions and measures, see also [Sak19].*

Example 3.2.9. *If* $\mathrm{G} = \mathrm{GL}_n$ *and* $\rho = \mathrm{Std}$*, the Hankel transform is calculated by H. Jacquet [Jac03]. Let* N *be the strictly upper triangula matrices in* GL_n*,* A *the universal Cartan of* GL_n*, identified with the maximal torus consisting of all diagonal matrices, embedded in* $\mathrm{N}\backslash\mathrm{GL}_n /\!\!/ \mathrm{N}$ *via* $t \mapsto [wt]$*, where* w *is the anti-diagonal matrix whose entries on the anti-diagonal are all* 1*.*

Theorem 3.2.10 ([Jac03, Sak19]). *We have a commutative diagram*

$$\begin{tikzcd} \mathcal{D}(\mathrm{Mat}_n) \arrow[rr, "\mathcal{F}_{\mathrm{Std}}"] \arrow[d] & & \mathcal{D}(\mathrm{Mat}_n^\vee) \arrow[d] \\ \mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi) \arrow[rr, "\mathcal{H}_{\mathrm{Std}}"] & & \mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi) \end{tikzcd}$$

and $\mathcal{H}_{\mathrm{Std}}$ *is given by the following formula*

$$\mathcal{H}_{\mathrm{Std}} = \mathcal{F}^{\psi}_{-\epsilon_1^\vee,\frac{1}{2}} \circ \psi(-e^{-\alpha_1}) \circ \mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}} \circ \cdots \circ \psi(-e^{-\alpha_{n-1}}) \circ \mathcal{F}_{-\epsilon_n^\vee,\frac{1}{2}},$$

where $\epsilon_i^\vee$ *is the cocharacter of the* i*-th coordinate of the torus of diagonal elements and the factor* $\psi(-e^{-\alpha_i})$ *denotes multiplication by* ψ *composed with the minus of the value of the indicated root.*

Example 3.2.11. *When* $\mathrm{G} = \mathrm{PGL}_2 \times \mathbb{G}_m$ *and* $\rho = \mathrm{Sym}^2$*. Let* $\lambda_+^\vee, \lambda_0^\vee$ *and* $\lambda_-^\vee$ *be the three weights of* ρ*, where* $\lambda_+^\vee$ *is dominant and* $\lambda_-^\vee$ *is anti-dominant. Let* $\mathrm{N} := \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\right\}$*. We have the following Hankel transform calculated by Y. Sakellaridis in [Sak22b].*

Theorem 3.2.12 ([Sak22b]). *The Hankel transform*

$$\mathcal{H}_{\mathrm{Sym}^2} : \mathcal{D}^-_{L(\mathrm{Sym}^2,\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi) \to \mathcal{D}^-_{L((\mathrm{Sym}^2)^\vee,\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$$

is given by the formula

$$\mathcal{H}_{\mathrm{Sym}^2} = \lambda(\eta_{\zeta^2-4},\psi)^{-1}\mathcal{F}_{-\lambda_+^\vee,\frac{1}{2}} \circ \delta_{1-4\zeta^{-2}} \circ \eta_{\zeta^2-4} \circ \mathcal{F}_{-\lambda_0^\vee,\frac{1}{2}} \circ \eta_{\zeta^2-4} \circ \mathcal{F}_{-\lambda_-^\vee,\frac{1}{2}},$$

where ζ *denotes coordinate on* $\mathrm{N}\backslash\mathrm{SL}_2 /\!\!/ \mathrm{N}$*,* η_D *is the quadratic character associated to the quadratic field extension* $\mathrm{F}(\sqrt{D})$*, and* δ_a *is the operator of multiplicative translation by* a *under the* $\mathbb{G}_m$*-action on* $\mathrm{N}\backslash\mathrm{G} /\!\!/ \mathrm{N}$*.*

For more details, see [Sak22b].

Remark 3.2.13. *As noted in [Sak23a], a remarkable feature of these formulae is that, despite the non-abelian nature of the* L*-functions involved, they are expressed through a combination of abelian Fourier transforms and specific correction factors. This structure mirrors the transfer operators in Chapter 2. By decomposing local Hankel transforms* *into compositions of classical Fourier transforms with appropriate correction factors, it becomes feasible to derive the Poisson summation formula without explicit reference to the upstairs space. Once the Poisson summation formula is established at the level of orbital integrals, it can be integrated into the trace formula to derive the functional equation of automorphic L-functions, which is the key point of the remaining part of the thesis.*

3.2.1 Orbital Integrals and Hankel Transforms for Standard L-functions of GL_2

In this section, we analyze the properties of local orbital integrals and the Hankel transforms associated with standard L-functions of $\mathrm{G} = \mathrm{GL}_2$. We demonstrate how the explicit formula for the Hankel transform serves as a local-to-global bridge, providing the necessary irregular distributions to establish the global Poisson summation formula.

Definition 3.2.14. *Denote $\mathcal{D}(\mathrm{Mat}_2)$ the pull-back of Shcwartz half-densities on Mat_2 to GL_2 via the natural embedding $g \mapsto g$. Note that a Schwartz half-density on Mat_2 is of the form $\Phi(x)(\mathrm{d}^+x)^{\frac{1}{2}}$, where $\Phi(x)$ is a usual Schwartz function on Mat_2 and d^+x is an additive Haar measure on Mat_2. Then its pull-back to GL_2 is $|\det g|\Phi(g)(\mathrm{d}g)^{\frac{1}{2}}$, where $\mathrm{d}g = \dfrac{\mathrm{d}^+g}{|\det g|^2}$ is the induced Haar measure on G. When there is no confusion, we will not distinguish the Schwartz half-density on Mat_2 and its pull-back to G.*

Remark 3.2.15. *Here we consider the natural action of $\mathrm{G}\times\mathrm{G}$ on both G and Mat_2 by*

$$x.(g_1, g_2) := g_1^{-1} x g_2,$$

for $x \in \mathrm{G}$ or Mat_2 and $(g_1, g_2) \in \mathrm{G}$. Later, we will consider another embedding of G into Mat_2 via the inverse map and modify the G-action.

Definition 3.2.16. *For a smooth half-density $\Psi(g)(\mathrm{d}g)^{\frac{1}{2}}$ on G, we define its twisted push-forward to the open subset T of $\mathfrak{C}$ as*

$$\delta^{\frac{1}{2}}(t)\mathcal{O}_t^{\psi}(\Psi)(\mathrm{d}t)^{\frac{1}{2}},$$

where for $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, $\delta(t) = \left|\dfrac{t_1}{t_2}\right|$,

$$\mathcal{O}_t^{\psi}(\Psi) = \int_{\mathrm{N}\times\mathrm{N}} \Psi(n_1 w t n_2)\psi^{-1}(n_1 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,$$

and $\mathrm{d}t$ *is the Haar measure on the torus* T *such that*

$$\int_{\mathrm{G}} \Psi(g)\,\mathrm{d}g = \int_{\mathrm{T}} \delta(t)\mathcal{O}_t^{\mathrm{triv}}(\Psi)\,\mathrm{d}t, \tag{3.1}$$

whenever this integral converges.

Remark 3.2.17. *In the following, we fix an additive character* $\psi : F \to \mathbb{C}$*, and choose the Haar measure on* Mat_2 *to be* $\mathrm{d}^+x := \mathrm{d}x_{11}\,\mathrm{d}x_{12}\,\mathrm{d}_{21}\,\mathrm{d}x_{22}$ *for* $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ *with each* $\mathrm{d}x_{ij}$ *being the self-dual Haar measure on the additive group* F *with respect to* ψ*. Then we choose* $\mathrm{d}g := \dfrac{\mathrm{d}^+g}{|\det g|^2}$*. We also choose the Haar measure on* N *to be the self-dual Haar measure with respect to* ψ*. Once all of these measures are chosen, the Haar measure* $\mathrm{d}t$ *on the torus* T *is the one such that the integration formula (3.1) holds. Using these choices, when* ψ *is unramified, we have* $\int_{\mathrm{T}(\mathfrak{o}_F)} 1\,\mathrm{d}t = \dfrac{1}{(1-q^{-1})^2}$*. Note that the normalization in (2.4) is such that the volume of* $\mathrm{G}(\mathfrak{o}_{\mathrm{F}}) = 1$*, but here we have*

$$\int_{\mathrm{G}(\mathfrak{o}_{\mathrm{F}})} 1\,\mathrm{d}g = \frac{1}{(1-q^{-2})(1-q^{-1})},$$

hence (2.4) is compatible with (3.1).

Definition 3.2.18. *The image of the twisted push-forward of* $\mathcal{D}(\mathrm{Mat}_2)$ *is denoted by*

$$\mathcal{D}^{-}_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi).$$

More explicitly, for $\Phi \in \mathcal{F}(\mathrm{Mat}_2)$*, the twisted push-forward of* $|\det g|\Phi(g)(\,\mathrm{d}g)^{\frac{1}{2}}$ *to* T *is*

$$|\det t|\delta^{\frac{1}{2}}(t)\mathcal{O}_t^{\psi}(\Phi)(\,\mathrm{d}t)^{\frac{1}{2}}.$$

Definition 3.2.19. *We define the basic vector* f° *to be the twisted push-forward of*

$$1_{\mathrm{Mat}_2(\mathfrak{o}_F)}(x)(\,\mathrm{d}^+x)^{\frac{1}{2}},$$

where $1_{\mathrm{Mat}_2(\mathfrak{o}_F)}$ *is the characteristic function of* $\mathrm{Mat}_2(\mathfrak{o}_F)$*.*

Proposition 3.2.20. *The restriction of* f° *to* $\mathrm{T}(\mathfrak{o}_F)$ *is* $(\,\mathrm{d}t)^{\frac{1}{2}}$ *in the case that* ψ *is unramified.*

Proof. Note that we have

$$\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & t_2 \\ t_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_1 n_1 & t_1 n_1 n_2 + t_2 \\ t_1 & t_1 n_2 \end{pmatrix}.$$

So in the case that $t_1, t_2 \in \mathfrak{o}_F^\times$,

$$\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & t_2 \\ t_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_2(\mathfrak{o}_F)$$

if and only if $n_1, n_2 \in \mathfrak{o}_F$. So we have $\mathcal{O}_t^\psi(1_{\mathrm{Mat}_2(\mathfrak{o}_F)}) = 1$ under the above conditions when $t \in \mathrm{T}(\mathfrak{o}_F)$. Note that $|\det t| = \delta^{\frac{1}{2}}(t) = 1$ as well in such a case, so $f^\circ|_{\mathrm{T}(\mathfrak{o}_F)} = (\,\mathrm{d}t)^{\frac{1}{2}}$. □

Proposition 3.2.21. *Let $f \in \mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi)$, then for fixed $t_1 \neq 0$,*

$$\frac{f(t_1,\cdot)}{|\det t|\delta^{\frac{1}{2}}(t)(\,\mathrm{d}t)^{\frac{1}{2}}}$$

is of Schwartz type when $t_2 \to \infty$, and can be extended across 0 smoothly. The extended function on F is a usual Schwartz function on F, where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. Moreover, when F is non-Archimedean and $t_1 \in \mathfrak{o}_F^\times$,

$$\frac{f^\circ(t_1,\cdot)}{|\det t|\delta^{\frac{1}{2}}(t)(\,\mathrm{d}t)^{\frac{1}{2}}} = 1_{\mathfrak{o}_F}(\cdot).$$

Proof. Assume f is the twisted push-forward of $\Phi(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}$, where $\Phi \in \mathcal{F}(\mathrm{Mat}_2)$, then

$$\begin{aligned}
&\frac{f}{|\det t|\delta^{\frac{1}{2}}(t)(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,t_2) \\
&= \int_{F\times F} \Phi\left(\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & t_2 \\ t_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.
\end{aligned}$$

Note that

$$\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & t_2 \\ t_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_1 n_1 & t_1 n_1 n_2 + t_2 \\ t_1 & t_1 n_2 \end{pmatrix},$$

for fixed $t_1 \in F^\times$, since the function Φ is of Schwartz type on Mat_2, by looking at the $(1,2)$-entry and the $(2,1)$-entry, this integral can be bounded by an integrable function that is independent of t_2, then by the dominant convergence theorem, we have

$$\int_{F\times F} \Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & t_2\\ t_1 & 0\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$\to \int_{F\times F} \Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ t_1 & 0\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,\ t_2\to 0.$$

When $t_2 \to \infty$, when F is non-Archimedean, when t_1n_1 and t_2n_2 are both in some compact set, $t_2 + t_1n_1n_2 \to \infty$ as $t_2 \to \infty$ since $t_1 \in F^\times$ is fixed, which means this integral is zero. The second claim is clear.

When F is Archimedean, since for any differential operator with polynomial coefficient D, we have DΦ is still a Schwartz function, then by dominant convergence theorem again we see $\dfrac{f(t_1,\cdot)}{|\det t|\delta^{\frac12}(t)(\,\mathrm{d}t)^{\frac12}}$ is a smooth function near 0. When t_2 is near ∞, since

$$\int_{|n_1|^2+|n_2|^2\gg 1}\left|\Phi\left(\begin{pmatrix}n_1 & \frac{n_1n_2}{t_1}+t_2\\ t_1 & n_2\end{pmatrix}\right)\right|\mathrm{d}n_1\,\mathrm{d}n_2 \ll 1.$$

and

$$\int_{|n_1|^2+|n_2|^2\ll 1}\left|\Phi\left(\begin{pmatrix}n_1 & \frac{n_1n_2}{t_1}+t_2\\ t_1 & n_2\end{pmatrix}\right)\right|\mathrm{d}n_1\,\mathrm{d}n_2 \ll \left|\frac{n_1n_2}{t_1}+t_2\right|^{-N} \ll |1+t_2|^{-N},$$

then

$$\left|\frac{f(t_1,\cdot)}{|\det t|\delta^{\frac12}(t)(\,\mathrm{d}t)^{\frac12}}\right| \ll |1+t_2|^{-N}.$$

Since again the derivatives of Φ is still a Schwartz function, we see all the derivatives of

$$\frac{f(t_1,\cdot)}{|\det t|\delta^{\frac12}(t)(\,\mathrm{d}t)^{\frac12}}$$

are of rapid decay at ∞, i.e., $\dfrac{f(t_1,\cdot)}{|\det t|\delta^{\frac12}(t)(\,\mathrm{d}t)^{\frac12}}$ is a Schwatz function. $\square$

Motivated by the Hankel transform calculated by [Jac03] and the global application, let us introduce the following intermediate half-densities and normalized intermediate functions:

Definition 3.2.22. *Let $f \in \mathcal{D}^{-}_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$, we define its intermediate half-density as*

$$\varphi_f := \psi(-e^{\alpha_1}) \circ \mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f),$$

where $\psi(-e^{\alpha_1})$ is the multiplication $\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi\left(-\frac{t_2}{t_1}\right)$. We define the normalized intermediate function

$$\widetilde{\varphi_f}(t) := \frac{\varphi_f(t_1,t_2)}{\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}}(\,\mathrm{d}t)^{\frac{1}{2}}},$$

where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$.

Lemma 3.2.23. *If we write $f = |\det t|\delta^{\frac{1}{2}}(t)\mathcal{O}_t(\Phi)$, then*

$$\widetilde{\varphi_f}(t_1,t_2) = |t_1|\psi\left(-\frac{t_2}{t_1}\right)\mathbb{F}_{\psi,2}(\mathcal{O}^{\psi}_{(t_1,\cdot)}(\Phi))\left(-\frac{1}{t_2}\right),$$

where $\mathbb{F}_{\psi,2}$ is the Fourier transform with respect to the second variable, which makes sense according to Proposition 3.2.21.

Proof. Write $f = |t_1|^{\frac{3}{2}}|t_2|^{\frac{1}{2}}\mathcal{O}^{\psi}_{\Phi}(t_1,t_2)(\,\mathrm{d}t)^{\frac{1}{2}}$ with $\Phi \in \mathcal{F}(\mathrm{Mat}_2)$. Then we have

$$\begin{aligned}\mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f)(t_1,t_2) &= \int_{F^\times} f(t_1,t_2x)\psi(x)|x|^{\frac{1}{2}}\,\mathrm{d}^\times x \\ &= \int_F |t_1|^{\frac{3}{2}}|t_2x|^{\frac{1}{2}}\mathcal{O}^{\psi}_{\Phi}(t_1,t_2x)(\,\mathrm{d}t)^{\frac{1}{2}}\psi(x)|x|^{\frac{1}{2}}\,\mathrm{d}^\times x.\end{aligned}$$

By making a change of variable $x \mapsto t_2^{-1}x$, we get

$$\frac{\mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f)(t_1,t_2)}{(\,\mathrm{d}t)^{\frac{1}{2}}} = |t_1|^{\frac{3}{2}}|t_2|^{-\frac{1}{2}}\int_F \mathcal{O}^{\psi}_{\Phi}(t_1,x)\psi(t_2^{-1}x)\,\mathrm{d}x.$$

□

Proposition 3.2.24. *For fixed $t_2 \in F^\times$, $\widetilde{\varphi_f}(\cdot,t_2)$ is of Schwartz type near ∞, and*

$$\widetilde{\varphi_f}(0,t_2) := \lim_{t_1\to 0}\widetilde{\varphi_f}(t_1,t_2)$$

exists. So for fixed $t_2 \in F^\times$, $\widetilde{\varphi_f}(\cdot,t_2)$ is a Schwartz function on F. Moreover, when $t_2 \in \mathfrak{o}_F^\times$ and ψ is unramified,

$$\widetilde{\varphi_{f^\circ}}(\cdot,t_2) = 1_{\mathfrak{o}_F}(\cdot).$$

Proof. Let us expand $\mathcal{O}^{\psi}_{\Phi}$, then $\mathbb{F}_{\psi,2}(\mathcal{O}^{\psi}_{(t_1,\cdot)}(\Phi))\left(-\frac{1}{t_2}\right)$ is

$$\int_F\int_{F\times F}\Phi\left(\begin{pmatrix} t_1n_1 & x+t_1n_1n_2 \\ t_1 & t_1n_2\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\cdot\psi(t_2^{-1}x)\,\mathrm{d}x.$$

Let $x+t_1n_1n_2\mapsto x$, $t_1n_1\mapsto n_1$ and $t_1n_2\mapsto n_2$, we have the above is

$$\int_{F^3}\Phi\left(\begin{pmatrix} n_1 & x \\ t_1 & n_2\end{pmatrix}\right)\psi^{-1}\left(\frac{n_1+n_2}{t_1}\right)\psi\left(\frac{xt_1-n_1n_2}{t_1t_2}\right)|t_1|^{-2}\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}x.$$

Therefore we have

$$\begin{aligned}
&\frac{\mathcal{F}^{\psi}_{-\epsilon_2^{\vee},\frac{1}{2}}(f)(t_1,t_2)}{(\,\mathrm{d}t)^{\frac{1}{2}}} && (3.2)\\
&=|t_1t_2|^{-\frac{1}{2}}\cdot\int_{F^3}\Phi\left(\begin{pmatrix} n_1 & x \\ t_1 & n_2\end{pmatrix}\right)\psi\left(\frac{x}{t_2}\right)\psi\left(-\frac{n_1+n_2}{t_1}-\frac{n_1n_2}{t_1t_2}\right)\mathrm{d}x\,\mathrm{d}n_1\,\mathrm{d}n_2 && (3.3)\\
&=|t_1t_2|^{-\frac{1}{2}}\int_F\mathbb{F}_{\psi,1}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} \frac{1}{t_1}+\frac{n_2}{t_1t_2} & -\frac{1}{t_2} \\ t_1 & n_2\end{pmatrix}\right)\psi\left(-\frac{n_2}{t_1}\right)\mathrm{d}n_2. && (3.4)
\end{aligned}$$

Make a change of variable that $n_2=t_1t_2z-t_2$, then we have

$$\frac{\mathcal{F}^{\psi}_{-\epsilon_2^{\vee},\frac{1}{2}}(f)(t_1,t_2)}{(\,\mathrm{d}t)^{\frac{1}{2}}}=|t_1t_2|^{\frac{1}{2}}\int_F\mathbb{F}_{\psi,1}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} z & -\frac{1}{t_2} \\ t_1 & t_1t_2z-t_2\end{pmatrix}\right)\psi\left(\frac{t_2}{t_1}-t_2z\right)\mathrm{d}z.$$

Therefore,

$$\widetilde{\varphi_f}(t_1,t_2)=|t_2|\int_F\mathbb{F}_{\psi,1}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} z & -\frac{1}{t_2} \\ t_1 & t_1t_2z-t_2\end{pmatrix}\right)\psi\left(-t_2z\right)\mathrm{d}z.$$

Then $\widetilde{\varphi_f}(\cdot,t_2)$ is a Schwartz function of t_1 for fixed $t_2\in F^{\times}$.

When F is non-Archimedean, $t_2\in\mathfrak{o}^{\times}$ and ψ is unramified, we can choose

$$\Phi_{11}=\Phi_{12}=\Phi_{21}=\Phi_{22}=1_{\mathfrak{o}_F},$$

then we have

$$\widetilde{\varphi_{f^{\circ}}}(t_1,t_2)=1_{\mathrm{Mat}_2(\mathfrak{o}_F)}(t_1)\int_F 1_{\mathrm{Mat}_2(\mathfrak{o}_F)}(z)1_{\mathrm{Mat}_2(\mathfrak{o}_F)}(t_1z)\psi^{-1}(z)\,\mathrm{d}z=1_{\mathfrak{o}_F}(t_1).$$

$\square$

Let $\mathrm{Mat}_2^\vee$ be the dual space of Mat_2, which can be identified with Mat_2 via the trace pairing. But we embed $G \hookrightarrow \mathrm{Mat}_2^\vee$ via the inverse map $g \mapsto g^{-1}$, and the action is given by $X(g_1, g_2) := g_2^{-1} X g_1$. Then $G \hookrightarrow \mathrm{Mat}_2^\vee$ is $G \times G$-equivariant.

Definition 3.2.25. *Let $\mathcal{D}(\mathrm{Mat}_2^\vee)$ be the pull-back of Schwartz half-densities of $\mathrm{Mat}_2^\vee$ to G via the above embedding. In this case, for $\Phi \in \mathcal{S}(\mathrm{Mat}_2^\vee)$, the pull-back of $\Phi(x)(\,\mathrm{d}x)^{\frac{1}{2}}$ is*

$$|\det g|^{-1}\Phi(g^{-1})(\,\mathrm{d}g)^{\frac{1}{2}}.$$

We use $\mathcal{D}^-_{L\left(\mathrm{Std}^\vee, \frac{1}{2}\right)}(\mathrm{N}, \psi\backslash \mathrm{G}/\mathrm{N}, \psi)$ to be the image of the twisted push-forward of $\mathcal{D}(\mathrm{Mat}_2^\vee)$ to the open subset T of $\mathfrak{C}$. More explicitly, the twisted push-forward of

$$|\det g|^{-1}\Phi(g^{-1})(\,\mathrm{d}g)^{\frac{1}{2}}$$

is

$$|\det t|^{-1}\delta^{\frac{1}{2}}(t)\int_{\mathrm{N}\times\mathrm{N}} \Phi((n_1 w t n_2)^{-1})\psi^{-1}(n_1 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \cdot (\,\mathrm{d}t)^{\frac{1}{2}} = |\det t|^{-1}\delta^{\frac{1}{2}}(t)\mathcal{O}^{\psi^{-1}}_{wt^{-1}w}(\Phi)(\,\mathrm{d}t)^{\frac{1}{2}}.$$

Definition 3.2.26. *We write $f^{\circ,\vee}$ for the basic vector that is defined to be the twisted push-forward of $1_{\mathrm{Mat}_2^\vee(\mathfrak{o}_F)}(\,\mathrm{d}x)^{\frac{1}{2}} \in \mathcal{S}(\mathrm{Mat}_2^\vee)$.*

Similar to the above case, we have

Proposition 3.2.27. *The restriction of $f^{\circ,\vee}$ to $\mathrm{T}(\mathfrak{o}_F)$ is $(\,\mathrm{d}t)^{\frac{1}{2}}$ in the case that ψ is unramified.*

We also have the exact proofs of the parallel version of Proposition 3.2.21 and Proposition 3.2.24.

Proposition 3.2.28. *For fixed $t_2 \neq 0$,*

$$\frac{f((\cdot)^{-1}, t_2)}{|\det t|^{-1}\delta^{\frac{1}{2}}(t)(\,\mathrm{d}t)^{\frac{1}{2}}}$$

can be extended to a Schwartz function on F, where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$.

Definition 3.2.29. *Let $f \in \mathcal{D}^-_{L\left(\mathrm{Std}^\vee, \frac{1}{2}\right)}(\mathrm{N}, \psi\backslash \mathrm{G}/\mathrm{N}, \psi)$, we define its intermediate half-density as*

$${}^\vee\varphi_f := \psi^{-1}(-e^{\alpha_1}) \circ \mathcal{F}_{\epsilon_1^\vee, \frac{1}{2}}(f),$$

where $\psi^{-1}(-e^{\alpha_1})$ *is still the multiplication* $\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \psi^{-1}\left(-\frac{t_2}{t_1}\right)$. *We define the normalized intermediate function.*

$$\widetilde{{}^{\vee}\varphi_f} := \frac{{}^{\vee}\varphi_f(t_1, t_2)}{\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}} (\,\mathrm{d}t)^{\frac{1}{2}}},$$

where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$.

Lemma 3.2.30. *If we write* $f = |\det t|^{-1}\delta^{\frac{1}{2}}(t)\mathcal{O}^{\psi^{-1}}_{wt^{-1}w}(\Phi)$, *then*

$$\widetilde{{}^{\vee}\varphi_f}(t_1, t_2) = |t_2|^{-1}\psi^{-1}\left(-\frac{t_2}{t_1}\right)\mathbb{F}_{\psi^{-1},2}(\mathcal{O}^{\psi^{-1}}_{(t_2^{-1},\cdot)}(\Phi))\,(-t_1)\,,$$

where $\mathbb{F}_{\psi,2}$ *is the Fourier transform with respect to the second variable, which makes sense according to Proposition 3.2.21.*

Proof. The proof is the same as Lemma 3.2.23. ◻

Proposition 3.2.31. *If we write* $f = |\det t|^{-1}\delta^{\frac{1}{2}}(t)\mathcal{O}^{\psi^{-1}}_{wt^{-1}w}(\Phi)$, *then for fixed* $t_1 \in F^{\times}$,

$$\widetilde{{}^{\vee}\varphi_f}(t_1, (\cdot)^{-1})$$

is of Schwartz type on F. *In particular,*

$$\widetilde{{}^{\vee}\varphi_f}(t_1, \infty) := \lim_{t_2\to\infty}\widetilde{{}^{\vee}\varphi_f}(t_1, t_2)$$

exists.

Proof. The proof is the same as Proposition 3.2.24. ◻

Consider the classical Fourier transform $\mathcal{F}(\mathrm{Mat}_2) \to \mathcal{F}(\mathrm{Mat}_2^{\vee})$ on the space of matrices:

$$\Phi(x) \mapsto \widehat{\Phi}(x),$$

where

$$\widehat{\Phi}(x) := \int_{\mathrm{Mat}_2}\Phi(y)\psi(\mathrm{tr}(yx))\,\mathrm{d}^{+}y.$$

Then $\Phi \mapsto \widehat{\Phi}$ is $\mathrm{G} \times \mathrm{G}$-equivariant with respect to the $\mathrm{G} \times \mathrm{G}$ actions on Mat_2 and $\mathrm{Mat}_2^\vee$ that we defined in the previous section. This Fourier transform induces a Fourier transform $\mathcal{F}$ on $\mathcal{D}(\mathrm{Mat}_2) \to \mathcal{D}(\mathrm{Mat}_2^\vee) : \Phi(x)(\,\mathrm{d}x)^{\frac{1}{2}} \mapsto \widehat{\Phi}(x)(\,\mathrm{d}x)^{\frac{1}{2}}$. This Fourier transform descends to the level of orbital integrals, and we have a beautiful formula to describe it on the level of orbital integrals.

Theorem 3.2.32 ([Jac03, Sak19])**.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{D}(\mathrm{Mat}_2) & \xrightarrow{\quad\mathcal{F}_{\mathrm{Std}}\quad} & \mathcal{D}(\mathrm{Mat}_2^\vee) \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi) & \xrightarrow{\mathcal{H}_{\mathrm{Std}}} & \mathcal{D}^-_{L\left(\mathrm{Std}^\vee,\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi) \end{array}$$

and $\mathcal{H}_{\mathrm{Std}}$ is given by the following formula

$$\mathcal{H}_{\mathrm{Std}} = \mathcal{F}^{\psi}_{-\epsilon_1^\vee,\frac{1}{2}} \circ \psi(-e^{-\alpha_1}) \circ \mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}.$$

Remark 3.2.33. *The calculation of this descent of the Fourier transform for general* GL_n *first appeared in [Jac03, Theorem 1], and was written is the above form in [Sak19, Theorem 8.1]. For the convenience of later reference and further comparison with the notations in [Jac03], let us copy the proof of [Sak19, Theorem 8.1] here.*

Proof. Let $\Phi \in \mathcal{F}(\mathrm{Mat}_2)$, recall that [Jac03] used the notation

$$\Omega(\Phi, \psi^{-1}, t) := \int_{\mathrm{N}\times\mathrm{N}} \Phi({}^t n_1 t n_2)\psi^{-1}(n_1 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

to denote the orbital integral. Note that ${}^t n_1 = wtw$, so we have $\Omega(\Phi, \psi^{-1} : t) = \mathcal{O}_{\mathrm{L}(w)\Phi}(t)$. And in [Jac03], the normalization is

$$\widetilde{\Omega}(\Phi, \psi^{-1} : t) := |t_1|\Omega(\Phi, \psi^{-1} : t),$$

so if we write $f(t)$ for the twisted push-forward of $|\det g|\Phi(g)(\,\mathrm{d}g)^{\frac{1}{2}}$, then we have $f(t) = |t_1 t_2|^{\frac{1}{2}}\widetilde{\Omega}(\mathrm{L}(w)\Phi, \psi^{-1} : t)(\,\mathrm{d}t)^{\frac{1}{2}}$.

Since the Fourier transform of $|\det g|\Phi(g)(\,\mathrm{d}g)^{\frac{1}{2}}$ is $\widehat{\Phi}(x)(\,\mathrm{d}x)^{\frac{1}{2}}$ on $\mathrm{Mat}_2^\vee$, whose pull-back to G is $|\det g|^{-1}\widehat{\Phi}(g^{-1})(\,\mathrm{d}g)^{\frac{1}{2}}$. Then, its twisted push-forward is

$$|\det t|^{-1}\delta^{1/2}(t)\int_{\mathrm{N}\times\mathrm{N}} \widehat{\Phi}(n_1 w \cdot wt^{-1}w \cdot n_2)\psi(n_1 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \cdot (\,\mathrm{d}t)^{\frac{1}{2}},$$

which is

$$|\det t|^{-1}\delta^{\frac{1}{2}}(t)\int_{\mathrm{N}\times\mathrm{N}}\widehat{\Phi}(w^t n_1 w t^{-1} w n_2)\psi(n_1 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\cdot(\,\mathrm{d}t)^{\frac{1}{2}},$$

and this is $|t_1t_2|^{-\frac{1}{2}}\widetilde{\Omega}(\mathrm{L}(w)\widehat{\Phi},\psi,wt^{-1}w)(\,\mathrm{d}t)^{\frac{1}{2}}$. Note that we have $\mathrm{L}(w)\widehat{\Phi}=(\mathrm{L}(w)\Phi)^\vee$ using the notation of [Jac03]. Then we have

$$\begin{aligned}
&\mathcal{F}^{\psi}_{-\epsilon_1^\vee,\frac{1}{2}}\circ\psi(-e^{-\alpha_1})\circ\mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f)(t_1,t_2)\\
&=\mathcal{F}^{\psi}_{-\epsilon_1^\vee,\frac{1}{2}}\left((b_1,b_2)\mapsto\psi\left(-\frac{b_2}{b_1}\right)\mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f)(b_1,b_2)\right)\\
&=\int_{F^\times}|p_1|^{\frac{1}{2}}\psi(p_1)\psi\left(-\frac{t_2}{p_1t_1}\right)\mathcal{F}^{\psi}_{-\epsilon_2^\vee,\frac{1}{2}}(f)(p_1t_1,t_2)\,\mathrm{d}^\times p_1\\
&=\int_{F^\times}|p_1|^{\frac{1}{2}}\psi(p_1)\psi\left(-\frac{t_2}{p_1t_1}\right)\int_{F^\times}|p_2|^{\frac{1}{2}}\psi(p_2)f(p_1t_1,p_2t_2)\,\mathrm{d}^\times p_2\,\mathrm{d}^\times p_1\\
&=\int_{F^\times}\int_{F^\times}|p_1p_2|^{\frac{1}{2}}\psi\left(p_1+p_2-\frac{t_2}{p_1t_1}\right)f(p_1t_1,p_2t_2)\,\mathrm{d}^\times p_2\,\mathrm{d}^\times p_1.
\end{aligned}$$

Make use of the relation that $f(t)=|t_1t_2|^{\frac{1}{2}}\widetilde{\Omega}((w,1)\Phi,\psi^{-1}:t)(\,\mathrm{d}t)^{\frac{1}{2}}$, we have the above is

$$\begin{aligned}
&\int_F\int_F|p_1p_2|^{\frac{1}{2}}\psi\left(p_1+p_2-\frac{t_2}{p_1t_1}\right)|p_1t_1p_2t_2|^{\frac{1}{2}}\widetilde{\Omega}\left((w,1)\Phi,\psi^{-1}:\begin{pmatrix}p_1t_1&0\\0&p_2t_2\end{pmatrix}\right)\\
&\cdot(\,\mathrm{d}p_1t_1)^{\frac{1}{2}}(\,\mathrm{d}p_2t_2)^{\frac{1}{2}}\,\mathrm{d}^\times p_2\,\mathrm{d}^\times p_1\\
&=\int_F\int_F|t_1t_2|^{\frac{1}{2}}\psi\left(p_1+p_2-\frac{t_2}{p_1t_1}\right)\widetilde{\Omega}\left((w,1)\Phi,\psi^{-1}:\begin{pmatrix}p_1t_1&0\\0&p_2t_2\end{pmatrix}\right)\,\mathrm{d}p_2\,\mathrm{d}p_1\cdot(\,\mathrm{d}t_1)^{\frac{1}{2}}(\,\mathrm{d}t_2)^{\frac{1}{2}}\\
&=|t_1t_2|^{-\frac{1}{2}}\int_F\int_F\psi\left(\frac{p_1}{t_1}+\frac{p_2}{t_2}-\frac{t_2}{p_1}\right)\widetilde{\Omega}\left((w,1)\Phi,\psi^{-1}:\begin{pmatrix}p_1&0\\0&p_2\end{pmatrix}\right)\,\mathrm{d}p_2\,\mathrm{d}p_1(\,\mathrm{d}t_1)^{\frac{1}{2}}(\,\mathrm{d}t_2)^{\frac{1}{2}}\\
&=|t_1t_2|^{-\frac{1}{2}}\widetilde{\Omega}\left((\mathrm{L}(w)\Phi)^\vee,\psi:\begin{pmatrix}t_2^{-1}&0\\0&t_1^{-1}\end{pmatrix}\right)(\,\mathrm{d}t)^{\frac{1}{2}}\\
&=\mathcal{H}_{\mathrm{Std}}(f),
\end{aligned}$$

where the last but two equality is [Jac03, Theorem 1]. □

Remark 3.2.34. *Using the notation in Proposition 3.2.22, we have*

$$\mathcal{H}_{\mathrm{Std}}(f)=\mathcal{F}^{\psi}_{-\epsilon_1^\vee,\frac{1}{2}}(\varphi_f).$$

More explicitly, we have

$$\frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,t_2) = |t_1t_2|^{-\frac{1}{2}}\int_F \widetilde{\varphi_\nu}(x,t_2)\psi\left(\frac{x}{t_1}\right)\,\mathrm{d}x$$
$$= |t_1t_2|^{-\frac{1}{2}}\mathbb{F}_{\psi,1}(\widetilde{\varphi_f}(\cdot,t_2))\left(-\frac{1}{t_1}\right).$$

Also note that $\mathcal{H}_{\mathrm{Std}}^{-1} = \mathcal{F}^{\psi^{-1}}_{\epsilon_2^\vee,\frac{1}{2}}\circ\psi^{-1}(-e^{-\alpha_1})\circ\mathcal{F}^{\psi^{-1}}_{\epsilon_1^\vee,\frac{1}{2}}$*, then for*

$$f\in\mathcal{D}^-_{L\left(\mathrm{Std}^\vee,\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi),$$

we have

$$\mathcal{H}_{\mathrm{Std}}^{-1}(f) = \mathcal{F}^{\psi^{-1}}_{\epsilon_2^\vee,\frac{1}{2}}({}^\vee\varphi_f).$$

and

$$\frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,t_2) = |t_1t_2|^{\frac{1}{2}}\int_F \widetilde{{}^\vee\varphi_f}(t_1,x^{-1})\psi^{-1}(xt_2)\,\mathrm{d}x$$
$$= |t_1t_2|^{\frac{1}{2}}\mathbb{F}_{\psi^{-1},2}(\widetilde{{}^\vee\varphi_f}(t_1,(\cdot)^{-1}))\,(-t_2)\,.$$

Let $\mathcal{H} := \mathcal{H}(\mathrm{G},\mathrm{G}(\mathfrak{o}))$ be the spherical Hecke algebra, which acts on $\mathrm{G}(\mathfrak{o})$-invariant functions on G by convolution:

$$h\star f(g) := \int_{\mathrm{G}} h(xg^{-1})f(g)\,\mathrm{d}g,\ h\in\mathcal{H}, g\in\mathrm{G}.$$

Then to isolate global spectrum in Section 3.4.5, we need the following fundamental lemma for spherical Hecke algebra:

Proposition 3.2.35. *Let* Φ *be a Schwartz function on* Mat_2*, then for* $h\in\mathcal{H}$*, we have have*

$$\widehat{h\star\Phi} = (h(\cdot)|\det\cdot|^2)\star\widehat{\Phi}$$

as functions on G*. In particular, if* ψ *is unramified, we have*

$$h\star\widehat{1_{\mathrm{Mat}_2(\mathfrak{o})}} = (h(\cdot)|\det\cdot|^2)\star 1_{\mathrm{Mat}_2(\mathfrak{o})}.$$

Proof.

$$\begin{aligned}\widehat{h\star\Phi}(x) &= \int_{\mathrm{Mat}_2} h\star\Phi(y)\psi(\mathrm{tr}(yx^{-1}))\,\mathrm{d}^+y\\
&= \int_{\mathrm{Mat}_2}\int_{\mathrm{G}} h(yg^{-1})\Phi(g)\,\mathrm{d}g\psi(\mathrm{tr}(yx^{-1}))\,\mathrm{d}^+y\\
&= \int_{\mathrm{Mat}_2}\int_{\mathrm{G}} h(g^{-1})\Phi(gy)\,\mathrm{d}g\psi(\mathrm{tr}(yx^{-1}))\,\mathrm{d}^+y\\
&= \int_{\mathrm{G}} h(g^{-1})\int_{\mathrm{Mat}_2}\Phi(y)\psi(\mathrm{tr}(yx^{-1}g^{-1}))|\det g|^{-2}\,\mathrm{d}^+y\\
&= \int_{\mathrm{G}} h(g^{-1})\widehat{\Phi}(gx)|\det g|^{-2}\,\mathrm{d}g\\
&= \int_{\mathrm{G}} h(xg^{-1})|\det xg^{-1}|^{2}\widehat{\Phi}(g)\,\mathrm{d}g.\end{aligned}$$

In particular, if ψ is unramified, we have $1_{\widehat{\mathrm{Mat}_2(\mathfrak{o})}} = 1_{\mathrm{Mat}_2}(\mathfrak{o})$, then

$$h\star\widehat{1_{\mathrm{Mat}_2(\mathfrak{o})}} = (h(\cdot)|\det\cdot|^2)\star 1_{\mathrm{Mat}_2(\mathfrak{o})}.$$

□

In the following, we will consider two types of irregular distributions on the space of orbital integrals, which will contribute to the boundary terms in the global Poisson summation formula.

Due to Proposition 3.2.21, we have the following definition

Definition 3.2.36. *We define the first type irregular distribution on*

$$\mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$$

as the following:

$$\mathcal{O}_{u,1}(f)(t_1) := \lim_{t_2\to 0}\frac{f}{|\det t|\delta^{\frac{1}{2}}(t)(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,t_2),\ f\in\mathcal{D}^-_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi),\ t_1\in F^\times,$$

where $t = \begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}$.

We have a parallel version for the dual side $\mathcal{D}^-_{L\left(\mathrm{Std}^\vee,\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$:

Definition 3.2.37. *We define the first type irregular distribution on*

$$\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi)$$

as the following:

$$\mathcal{O}^{\vee}_{u,1}(f)(t_2) := \lim_{t_1\to\infty} \frac{f}{|\det t|^{-1}\delta^{\frac{1}{2}}(t)(\mathrm{d}t)^{\frac{1}{2}}}(t_1,t_2),\ f\in \mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi),\ t_2\in F^{\times},$$

where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$.

We need the following easy lemma in Section 3.3.2:

Lemma 3.2.38. *If the half-density is of the form* $\mathcal{H}_{\mathrm{Std}}(f)$ *for some*

$$f\in \mathcal{D}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi),$$

then we have

$$\mathcal{O}^{\vee}_{u,1}(\mathcal{H}_{\mathrm{Std}}(f))(t_2) = |t_2|\mathbb{F}_{\psi,1}(\widetilde{\varphi_f}(\cdot,t_2))(0).$$

Proof. This follows from Remark 3.2.34 directly. □

Motived by Theorem 3.2.32 and Proposition 3.2.24, we have the following second type irregular distributions

Definition 3.2.39. *We define the second type irregular distribution on*

$$\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash \mathrm{G}/\mathrm{N},\psi)$$

in the following way:

$$\mathcal{O}_{u,2}(f)(t_2) := \widetilde{\varphi_f}(0,t_2), t_2\in F^{\times}.$$

To relate it to orbital integrals of the upstairs function, it turns out that it comes from the *central orbital integral*. Let $\Phi(x)$ be a Schwartz function on Mat_2, and $z\in F^{\times}$, consider the orbital integral

$$\phi_{\Phi}(z) := \int_{\mathrm{N}} \Phi(z\mathrm{I}_2\cdot n)\psi^{-1}(n)\,\mathrm{d}n,$$

where the notation indicates that it is related to the function ϕ used in [Jac03]. We have

Proposition 3.2.40.

$$\phi_\Phi(z) = |z|^{-2}\widetilde{\varphi_f}(0,-z),$$

where f is the twisted push forward of $\Phi(x)(\,\mathrm{d}x)^{\frac{1}{2}}$ and $\widetilde{\varphi_f}$ is the normalized intermediate function of f defined in Definition 3.2.22.

Proof. Using the notations in [Jac03], $\phi_\Phi(z) = \Omega((w,1)\Phi, \psi^{-1} : wz)$. According to [Jac03, Proposition 4], $\phi_\Phi(z)$ is a smooth function with compact support on $F^\times$, and is given by

$$\phi_\Phi(z) = |z|^{-3}\int_F \Omega\left(((w,1)\Phi)^\vee, \psi : \begin{pmatrix} -z^{-1} & 0 \\ 0 & t_1 \end{pmatrix}\right) \mathrm{d}t_1.$$

Note that $\Omega\left(((w,1)\Phi)^\vee, \psi : \begin{pmatrix} -z^{-1} & 0 \\ 0 & t_1 \end{pmatrix}\right) = \frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1^{-1},-z)|t_1|^{-\frac{1}{2}}|z|^{\frac{3}{2}}$, so the above integral is

$$\int_F \frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1^{-1},-z)|t_1|^{-\frac{1}{2}}|z|^{-\frac{3}{2}}\,\mathrm{d}t_1 = \int_F \frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,-z)|bz|^{-\frac{3}{2}}\,\mathrm{d}t_1.$$

According to Remark 3.2.34, we have

$$\begin{aligned}
\mathcal{H}_{\mathrm{Std}}(f)(t_1,-z) &= \mathcal{F}^{\psi^{-1}}_{\epsilon_2^\vee,\frac{1}{2}}(\varphi_f)(t_1,-z) \\
&= \int_F \widetilde{\varphi_f}(x,-z)\psi(t_1^{-1}x)\,\mathrm{d}x \cdot |t_1|^{-\frac{1}{2}}|z|^{-\frac{1}{2}}(\,\mathrm{d}t_1)^{\frac{1}{2}}(\,\mathrm{d}z)^{\frac{1}{2}} \\
&= \mathbb{F}_{\psi,1}(\widetilde{\varphi_f})(-t_1^{-1},-z)|t_1|^{-\frac{1}{2}}|z|^{-\frac{1}{2}}(\,\mathrm{d}t_1)^{\frac{1}{2}}(\,\mathrm{d}z)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_F \frac{\mathcal{H}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t_1,-z)|t_1 z|^{-\frac{3}{2}}\,\mathrm{d}t_1 \\
&= \int_F \mathbb{F}_{\psi,1}(\widetilde{\varphi_f})(-t_1^{-1},-z)|t_1|^{-2}|z|^{-2}\,\mathrm{d}t_1 \\
&= |z|^{-2}\widetilde{\varphi_f}(0,-z)
\end{aligned}$$

according to the Fourier inversion formula since $\widetilde{\varphi_f}(\cdot,z)$ is a Schwartz function on F for fixed $z \in F^\times$ due to Proposition 3.2.24.

∎

We also have the counterpart for the dual space $\mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$.

Definition 3.2.41. *We define the second type irregular distribution on*

$$\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$$

in the following way:

$$\mathcal{O}^{\vee}_{u,2}(f)(t_1) := \widetilde{{}^{\vee}\varphi_f}(t_1,\infty),\ f\in\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi), t_1\in F^{\times}.$$

Let $\Phi\in\mathcal{S}(\mathrm{Mat}_2^{\vee})$, consider the corresponding integral

$${}^{\vee}\phi_{\Phi}(z) := \int_{\mathrm{N}}\Phi(z^{-1}\mathrm{I}_2\cdot n)\psi(n)\,\mathrm{d}n,$$

we have

Proposition 3.2.42.

$${}^{\vee}\phi_{\Phi}(z) = |z|^2\cdot\widetilde{{}^{\vee}\varphi_f}(-z^{-1},\infty),$$

where f is the twisted push forward of $\Phi(x)(\,\mathrm{d}^{+}x)^{\frac{1}{2}}$, and $\widetilde{{}^{\vee}\varphi_f}$ is the normalized intermediate function of f defined in Definition 3.2.29.

Proof. Using the notations in [Jac03], $\phi^{\vee}_{\Phi}(z)=\Omega((w,1)\Phi,\psi:wz)$. According to [Jac03, Proposition 4] again, $\phi^{\vee}_{\Phi}(z)$ is a smooth function with compact support on $F^{\times}$, and is given by

$$\phi^{\vee}_{\Phi}(z) = |z|^3\int_F\Omega\left(((w,1)\Phi)^{\vee},\psi^{-1}:\begin{pmatrix}-z & \\ & t_2\end{pmatrix}\right)\mathrm{d}t_2.$$

Note that $\Omega\left(((w,1)\Phi)^{\vee},\psi^{-1}:\begin{pmatrix}-z & 0\\ 0 & t_2\end{pmatrix}\right) = \dfrac{\mathcal{H}^{-1}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(-z,t_2)|t_2|^{-\frac{1}{2}}|z|^{\frac{3}{2}}$, so the above integral is

$$\int_F\frac{\mathcal{H}^{-1}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(-z,t_2)|t_2|^{-\frac{1}{2}}|z|^{\frac{3}{2}}\,\mathrm{d}t_2 = \int_F\frac{\mathcal{H}^{-1}_{\mathrm{Std}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(-z,t_2^{-1})|t_2z^{-1}|^{-\frac{3}{2}}\,\mathrm{d}t_2. \tag{3.5}$$

According to Remark 3.2.34, we have

$$\begin{aligned}\mathcal{H}^{-1}_{\mathrm{Std}}(f)(-z,t_2^{-1}) &= \mathcal{F}_{\epsilon_2^{\vee},\frac{1}{2}}({}^{\vee}\varphi_f)(-z,t_2^{-1})\\ &= \int_F\widetilde{{}^{\vee}\varphi_f}(-z,x^{-1})\psi^{-1}(t_2^{-1}x)\,\mathrm{d}x\cdot|t_2|^{-\frac{1}{2}}|z|^{\frac{1}{2}}(\,\mathrm{d}t_2)^{\frac{1}{2}}(\,\mathrm{d}z)^{\frac{1}{2}}\\ &= \mathbb{F}_{\psi^{-1},2}(\widetilde{{}^{\vee}\varphi_f}(-z,(\cdot)^{-1}))(-t_2^{-1})|t_2|^{-\frac{1}{2}}|z|^{\frac{1}{2}}(\,\mathrm{d}t_2)^{\frac{1}{2}}(\,\mathrm{d}z)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_F \frac{\mathcal{H}_{\mathrm{Std}}^{-1}(f)}{(\mathrm{d}t)^{\frac{1}{2}}}(-z,t_2^{-1})|t_2z^{-1}|^{-\frac{3}{2}}\,\mathrm{d}t_2\\
&=\int_F \mathbb{F}_{\psi^{-1},2}({}^{\vee}\widetilde{\varphi_f}(-z,(\cdot)^{-1}))(-t_2^{-1})|t_2|^{-2}|z|^2\,\mathrm{d}t_2\\
&=|z|^2\,{}^{\vee}\widetilde{\varphi_f}(-z,\infty)
\end{aligned}$$

according to the Fourier inversion formula again. □

Let us conclude this section with another lemma that will be used in Section 3.3.2:

Lemma 3.2.43. *Let $f\in\mathcal{D}^{-}_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi)$, then we have*

$$\mathcal{O}^{\vee}_{u,2}(\mathcal{H}_{\mathrm{Std}}(f))(z)=|z|^3\mathbb{F}_{\psi,2}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0)$$

if f is the twisted push-forward of $\Phi(x)(\mathrm{d}^+x)^{\frac{1}{2}}$.

Proof. According to (3.5), in the case that f is the twisted push-forward of $\Phi(x)(\mathrm{d}^+x)^{\frac{1}{2}}$, we have

$$\mathcal{O}^{\vee}_{u,2}(\mathcal{H}_{\mathrm{Std}}(f))(z)=\int_F|-z|^{\frac{3}{2}}|t_2|^{\frac{1}{2}}\mathcal{O}_{\Phi}(-z,t_2)|t_2|^{-\frac{1}{2}}|z|^{\frac{3}{2}}\,\mathrm{d}t_2=|z|^3\mathbb{F}_{\psi,2}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0).$$

□

3.3 Global Theory: Poisson Summation Formulae

Returning to the framework of the Braverman-Kazhdan-Ngô proposal, the primary objective in utilizing the Kuznetsov trace formula is to establish the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{S}_{\rho}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi) \arrow[rr, "\mathcal{H}_\rho"] \arrow[dr, "\mathrm{KTF}"'] & & \mathcal{S}_{\rho^\vee}(\mathrm{N},\psi\backslash\mathrm{G}/\mathrm{N},\psi) \arrow[dl, "\mathrm{KTF}"] \\
& \mathbb{C} &
\end{tikzcd}.$$

This diagram illustrates the fundamental compatibility between the local Hankel transforms and the global Poisson summation formula, which ultimately will yield the functional equation for the associated L-functions.

While the expression of KTF with a non-standard test may diverge, we can introduce a parameter s to deform these measures, ensuring absolute convergence when $\mathrm{Re}(s) \gg 0$. A critical step in this process is utilizing the Poisson summation formula to provide control of the KTF within the non-convergence region. By applying the Phragmen-Lindelöf principle to obtain the full spectral decomposition, finally we should be able to establish the functional equation of the associated L-functions. In the following, we provide a detailed demonstration of this procedure for the case $\mathrm{G} = \mathrm{GL}_2$ and $\rho = \mathrm{Std}$.

Remark 3.3.1. *Instead of using trace formula, D. Jiang and Z. Luo [JL22, JL23] introduced Schwartz spaces $\mathcal{S}_\pi(\mathbb{A}^\times) \subset \mathcal{C}^\infty(\mathbb{A}^\times)$ and π-Fourier transforms $\mathcal{F}_{\pi,\psi} : \mathcal{S}_\pi(\mathbb{A}^\times) \to \mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$ with a non-trivial additive character ψ of $k\backslash\mathbb{A}$, proved the associated Poisson Summation Formula over $\mathbb{A}^\times$, which is related to GL_n via the determinant map, based on the Godement-Jacquet theory for the standard L-functions.*

While the most effective approach for establishing analytic properties of L-functions remains a subject of active debate, this framework has already yielded significant applications. In the adjoint work [JL25] with D. Jiang, we compared the functional equation given by Godement-Jacquet theory with the one from the Rankin-Selberg convolution, and found that the local Fourier transforms associated with irreducible representations are responsible for the Hankel transforms in the Voronoi Summation Formula. By using the global Poisson Summation Formula, we re-proved the Voronoi Summation Formula in [IT13]. For more details, see [JL25].

In the following, let k be a global field, and for each place ν of k, we set k_ν to be the completion of k at ν. Write $\mathfrak{o}_\nu$ and ϖ_ν for the ring of integers and a local uniformizer if ν is a finite place. Let $\mathbb{A}$ be the ring of adeles of k. We fix an adelic character $\psi = \otimes'_\nu \psi_\nu : \mathbb{A}/k \to \mathbb{C}$. To emphasize the place ν, we will use $\mathcal{D}(\mathrm{Mat}_{2,\nu})$, $\mathcal{D}(\mathrm{Mat}^\vee_{2,\nu})$, $\mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N}_\nu, \psi_\nu\backslash\mathrm{G}_\nu/\mathrm{N}_\nu, \psi_\nu)$, $\mathcal{D}^-_{L(\mathrm{Std}^\vee,\frac{1}{2})}(\mathrm{N}_\nu, \psi_\nu\backslash\mathrm{G}_\nu/\mathrm{N}_\nu, \psi_\nu)$ to denote the local spaces that we defined in Section 3.2.1 when $F = k_\nu$. We will use f°_ν and $f^{\circ,\vee}_\nu$ to denote the basic vectors, and $\mathcal{H}_{\mathrm{Std},\nu}$ to be the Hankel transform. We also use $\mathcal{O}_{u,1,\nu}$, $\mathcal{O}^\vee_{u,1,\nu}$, $\mathcal{O}_{u,2,\nu}$ and $\mathcal{O}^\vee_{u,2,\nu}$ to denote the irregular distributions. We will also use $\mathbb{F}_{\psi_\nu}, \mathbb{F}_{\psi_\nu,i}$ to denote the local Fourier transform with respect to ψ_ν, the partial Fourier transforms with the i-th variable, and will use $\mathbb{F}_\psi = \otimes'_\nu \mathbb{F}_{\psi_\nu}, \mathbb{F}_{\psi,i,\mathbb{A}} = \otimes'_\nu \mathbb{F}_{\psi_\nu,i}$ to denote their global versions.

Definition 3.3.2. *We define the global space of orbital integrals*

$$\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi):=\otimes'_{\nu}\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}_{\nu},\psi_{\nu}\backslash \mathrm{G}_{\nu}/\mathrm{N}_{\nu},\psi_{\nu}),$$

where the restricted tensor product is taken with respect to the basic vectors f°_{ν}. We define

$$\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$$

in a similar way.

Remark 3.3.3. *Thanks to Proposition 3.2.20, the restriction of f°_{ν} to $\mathrm{T}(\mathfrak{o}_{\nu})$ is $(\,\mathrm{d}t_{\nu})^{\frac{1}{2}}$ for almost all ν, then we can identify $\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$ as half-densities on $\mathrm{T}(\mathbb{A})$.*

According to Proposition 3.2.35, we can define a global Hankel transform

$$\mathcal{H}_{\mathrm{Std},\mathbb{A}}:=\otimes'\mathcal{H}_{\mathrm{Std},\nu}:\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)\to\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi).$$

Now, we state the global Poisson summation formula on the level of orbital integrals. Let $f\in\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$, we first define

Definition 3.3.4.

$$\mathrm{KTF}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(f):=\sum_{t\in\mathrm{T}(k)}\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)+\sum_{t_1\in k^{\times}}\mathcal{O}_{u,1,\mathbb{A}}(f)(t_1)+\sum_{t_2\in k^{\times}}\mathcal{O}_{u,2,\mathbb{A}}(f)(t_2),$$

where $\mathcal{O}_{u,1,\mathbb{A}}:=\otimes'_{\nu}\mathcal{O}_{u,1,\nu}$ and $\mathcal{O}_{u,2,\mathbb{A}}:=\otimes'_{\nu}\mathcal{O}_{u,2,\nu}$.

Similarly for $f\in\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$, we define

$$\mathrm{KTF}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(f):=\sum_{t\in\mathrm{T}(k)}\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)+\sum_{t_2\in k^{\times}}\mathcal{O}^{\vee}_{u,1,\mathbb{A}}(f)(t_2)+\sum_{t_1\in k^{\times}}\mathcal{O}^{\vee}_{u,2,\mathbb{A}}(f)(t_1),$$

where $\mathcal{O}^{\vee}_{u,1,\mathbb{A}}:=\otimes'_{\nu}\mathcal{O}^{\vee}_{u,1,\nu}$ and $\mathcal{O}_{u,2,\mathbb{A}}:=\otimes'_{\nu}\mathcal{O}^{\vee}_{u,2,\nu}$.

We need the following lemmas to guarantee the absolute convergence of the sums in the above Definition 3.3.4.

Lemma 3.3.5. *For*

$$f\in\mathcal{D}^{-}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)\ (\text{resp. }\mathcal{D}^{-}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)),$$

we have

(1) $\displaystyle\sum_{t\in \mathrm{T}(k)} \frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)$ *is absolutely convergent.*

(2) $\displaystyle\sum_{t_1\in k^\times} \mathcal{O}_{u,1,\mathbb{A}}(f)(t_1)$ *(resp.* $\displaystyle\sum_{t_2\in k^\times} \mathcal{O}^\vee_{u,1,\mathbb{A}}(f)(t_2)$*) is absolutely convergent.*

(3) $\displaystyle\sum_{t_2\in k^\times} \mathcal{O}_{u,2,\mathbb{A}}(f)(t_2)$ *(resp.* $\displaystyle\sum_{t_1\in k^\times} \mathcal{O}^\vee_{u,2,\mathbb{A}}(f)(t_1)$*) is absolutely convergent.*

Proof. We only prove the statements for $f \in \mathcal{D}^-_{L(\mathrm{Std},\frac{1}{2})}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$. Let Φ be a Schwartz function on $\mathrm{Mat}_2(\mathbb{A})$ such that f is the twisted push-forward of $\Phi(x)(\,\mathrm{d}x)^{\frac{1}{2}}$. Let us consider (1) and (2) together.

Note that we have

$$\begin{aligned}
&\sum_{t\in \mathrm{T}(k)} \left|\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)\right| + \sum_{t_1\in k^\times} \mathcal{O}_{u,1,\mathbb{A}}(f)(t_1)\\
&= \sum_{t_1\in k^\times, t_2\in k} \left| \int_{\mathrm{N}(\mathbb{A})\times\mathrm{N}(\mathbb{A})} \Phi\left(n_1 w \begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix} n_2\right) \psi^{-1}(n_1n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\right|\\
&\leq \sum_{t_1\in k^\times, t_2\in k} \int_{\mathrm{N}(\mathbb{A})\times\mathrm{N}(\mathbb{A})} \left|\Phi\left(\begin{pmatrix} t_1n_1 & t_1n_1n_2+t_2\\ t_1 & t_1n_2\end{pmatrix}\right)\right| \,\mathrm{d}n_1\,\mathrm{d}n_2.
\end{aligned}$$

We may assume Φ is positive and applying the Poisson summation formula, we have

$$\sum_{t_2\in k} \Phi\left(\begin{pmatrix} t_1n_1 & t_1n_1n_2+t_2\\ t_1 & t_1n_2\end{pmatrix}\right) = \sum_{t_2\in k} \mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} t_1n_1 & t_2\\ t_1 & t_1n_2\end{pmatrix}\right)\psi(t_1t_2n_1n_2),$$

which is bounded by

$$\sum_{t_2\in k} \left|\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} t_1n_1 & t_2\\ t_1 & t_1n_2\end{pmatrix}\right)\right|.$$

Therefore, (1) and (2) are proved.

As for (3), according to Proposition 3.2.40, we have

$$\begin{aligned}
\sum_{t_2\in k^\times} |\mathcal{O}_{u,2,\mathbb{A}}(f)(t_2)| &= \sum_{z\in k^\times} \left|\int_{\mathrm{N}(\mathbb{A})} \Phi(z\mathrm{I}_2\cdot n)\psi^{-1}(n)\,\mathrm{d}n\right|\\
&\leq \sum_{z\in k^\times} \int_{\mathrm{N}(\mathbb{A})} \left|\Phi\left(\begin{pmatrix} z & zn\\ & z\end{pmatrix}\right)\right| \,\mathrm{d}n,
\end{aligned}$$

which converges absolutely.

$\square$

Then we have the following Poisson summation formula

Theorem 3.3.6. *For* $f \in \mathcal{D}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}),\psi)$*, we have*

$$\mathrm{KTF}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(f) = \mathrm{KTF}_{L\left(\mathrm{Std}^{\vee},\frac{1}{2}\right)}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)).$$

3.3.1 Poisson Summation Formula for Standard L-functions of GL_2: An Indirect Proof

In this section, we give an indirect proof of the Poisson summation formula, which is based on the classical Poisson summation formula on the upstairs spaces $\mathrm{Mat}(\mathbb{A})$ and $\mathrm{Mat}^{\vee}(\mathbb{A})$. The more important thing in the view of this paper is the direct proof from the next section, and this section can be viewed as a double-check that Theorem 3.3.6 coincides with the classical one. The reader may also skip this section and move to the next section.

proof of Theorem 3.3.6. By linearity, we may assume $f = \otimes'_{\nu} f_{\nu}$, with each f_{ν} being the twisted push-forward of some $\Phi_{\nu}(x) \in \mathcal{S}(\mathrm{Mat}_{2,\nu})$. Let $\Phi = \otimes'_{\nu}\Phi_{\nu} \in \mathcal{S}(\mathrm{Mat}_2(\mathbb{A}))$. Consider the integral

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{Mat}_2(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

A first observation is that we have the double cosets decomposition

$$\mathrm{Mat}_2(k) = \mathrm{B}(k)w\mathrm{B}(k)\bigsqcup\mathrm{B}(k)\bigsqcup\{g\in\mathrm{Mat}_2(k)\mid g\neq 0, \det g = 0\}\bigsqcup\{0\}.$$

For the open Brahut cell, we have

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{B}(k)w\mathrm{B}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

$$=\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{T}(k)}\sum_{\delta_1\in\mathrm{N}(k)}\sum_{\delta_2\in\mathrm{N}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\delta_1 w\gamma\delta_2\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

$$=\int_{\mathbb{A}}\int_{\mathbb{A}}\sum_{\gamma\in\mathrm{T}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}w\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

$$=\sum_{\gamma\in\mathrm{T}(k)}\prod_{\nu}\int_{k_\nu}\int_{k_\nu}\Phi_\nu\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}w\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi_\nu^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

$$=\sum_{\gamma\in\mathrm{T}(k)}\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(\gamma).$$

For the sum over $\mathrm{B}(k)$, we have

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{B}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{T}(k)}\sum_{\delta\in\mathrm{N}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\delta\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{x\in\mathbb{A}}\int_{y\in k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{T}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\gamma^{-1}\cdot\gamma\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

Note that we have for $\gamma=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}$,

$$\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}^{-1}=\begin{pmatrix}1 & \gamma_1\gamma_2^{-1}n_2\\ 0 & 1\end{pmatrix},$$

and make a change of variable by $y\mapsto\gamma_1^{-1}\gamma_2 y$ and $x\mapsto x-y$, the above integral becomes

$$\int_{x\in\mathbb{A}}\int_{y\in k\backslash\mathbb{A}}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\right)\psi^{-1}(n_1-n_2+\gamma_1^{-1}\gamma_2 n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,$$

with the integral over y being non-zero only when $\gamma_1=\gamma_2$, in which case it is equal to

$$\sum_{\gamma\in\mathrm{Z}(k)}\int_{\mathbb{A}}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\right)\psi^{-1}(n_1)\,\mathrm{d}n_1,$$

where Z is the center of GL_2 consisting of scalar matrices. This inner integral admits an Euler product

$$\prod_{\nu}\int_{k_\nu}\Phi_\nu\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\gamma\right)\psi_\nu^{-1}(x)\,\mathrm{d}x,$$

which according to Proposition 3.2.40 is

$$\sum_{t_2\in k^\times}\mathcal{O}_{u,2,\mathbb{A}}(f)(t_2).$$

Next, for the rank 1 matrices, it turns out the only non-zero-sum is the following

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{0\neq\gamma=(\gamma_{ij}),\gamma_{12}\neq 0,\det\gamma=0}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

We can simplify the above sum, which is

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{t_1\in k^\times}\sum_{\delta_1\in\mathrm{N}(k)}\sum_{\delta_2\in\mathrm{N}(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\delta_1\begin{pmatrix}0 & 0\\ t_1 & 0\end{pmatrix}\delta_2\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$=\sum_{t_1\in k^\times}\int_{\mathbb{A}}\int_{\mathbb{A}}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ t_1 & 0\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,$$

Also, note that this sum is nothing but

$$\sum_{t_1\in k^\times}\mathcal{O}_{u,1,\mathbb{A}}(f)(t_1).$$

Then, the integral we start with is just

$$\sum_{t\in\mathrm{T}(k)}\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)+\sum_{t_1\in k^\times}\mathcal{O}_{u,1,\mathbb{A}}(f)(t_1)+\sum_{t_2\in k^\times}\mathcal{O}_{u,2,\mathbb{A}}(f)(t_2)=\mathrm{KTF}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(f).$$

For fixed $n_1,n_2\in\mathrm{N}(\mathbb{A})$, we have the usual Poisson summation formula

$$\sum_{\gamma\in\mathrm{Mat}_2(k)}\Phi(n_1\gamma n_2)=\sum_{\gamma\in\mathrm{Mat}_2^\vee(k)}\widehat{\Phi}(n_2^{-1}\gamma n_1^{-1}),$$

where $\widehat{\Phi}:=\otimes'_\nu\widehat{\Phi_\nu}$ is the Fourier transform of Φ. Then we have

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{Mat}_2(k)}\Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$=\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{Mat}^\vee(k)}\widehat{\Phi}\left(\begin{pmatrix}1 & -n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & -n_2\\ 0 & 1\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,$$

which is

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{Mat}^\vee(k)}\widehat{\Phi}\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi(n_1+n_2)\,\mathrm{d}x\,\mathrm{d}y$$

by changing n_1 to $-n_1$, and n_2 to $-n_2$. Similarly, we have

$$\mathrm{Mat}_2^\vee(k)=\mathrm{B}(k)w\mathrm{B}(k)\bigsqcup\mathrm{B}(k)\bigsqcup\{g\in\mathrm{Mat}_2^\vee(k)\mid g\neq 0,\det g=0\}\bigsqcup\{0\},$$

and the same computation shows that

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{B}(k)w\mathrm{B}(k)\subset\mathrm{Mat}_2^\vee}\widehat{\Phi}\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$=\sum_{t\in\mathrm{T}(k)}\frac{\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t),$$

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{\gamma\in\mathrm{B}(k)\subset\mathrm{Mat}_2^\vee}\widehat{\Phi}\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$=\sum_{t_1\in k^\times}\mathcal{O}^\vee_{u,2,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f))(t_1),$$

and

$$\int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\sum_{0\neq\gamma=(\gamma_{ij}),\gamma_{12}\neq 0,\det\gamma=0}\widehat{\Phi}\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$=\sum_{t_2\in k^\times}\mathcal{O}^\vee_{u,1,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f))(t_2).$$

Therefore, we obtain

$$\mathrm{KTF}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(f)=\mathrm{KTF}_{L\left(\mathrm{Std}^\vee,\frac{1}{2}\right)}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)).$$

□

3.3.2 Poisson Summation Formula for Standard L-functions of GL_2: A Direct Proof

In this section, we will use the local properties we established in Section 3.2.1 and a two-step classical one-dimensional Poisson Summation Formula to prove Theorem 3.3.6

directly. This is compatible with the local Hankel transform in the sense that the local Hankel transform is a composition of two Fourier transforms with some mysterious correction factors. The point of this section is that we can give a proof of Theorem 3.3.6 independent of the Poisson summation formula on the upstairs space, which has the potential to be generalized to other reductive groups and Hankel transforms. This may be used to establish analytic properties of general automorphic L-functions.

To begin with, let us assume $f = \otimes'_\nu f_\nu$. For each f_ν, recall that we have the intermediate half-densities and normalized intermediate functions defined in Definition 3.2.22:

$$\varphi_\nu := \psi_\nu(-e^{\alpha_1}) \circ \mathcal{F}^{\psi_\nu}_{-\epsilon_2^\vee, \frac{1}{2}}(f_\nu),$$

$$\widetilde{\varphi_\nu} = \frac{\varphi_\nu(t_1, t_2)}{|\frac{t_1}{t_2}|_\nu^{\frac{1}{2}} (\mathrm{d}t)^{\frac{1}{2}}},$$

and write $\widetilde{\varphi_f} := \otimes'_\nu \widetilde{\varphi_\nu}$.

Lemma 3.3.7. *For $f \in \mathcal{D}^-_{L(\mathrm{Std}, \frac{1}{2})}(\mathrm{N}(\mathbb{A}), \psi\backslash \mathrm{G}(\mathbb{A})/\mathrm{N}(\mathbb{A}), \psi)$, we have*

$$\sum_{t \in \mathrm{T}(k)} \frac{\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)}{(\mathrm{d}t)^{\frac{1}{2}}}(t) + \sum_{t_2 \in k^\times} \mathcal{O}^\vee_{u,1,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f))(t_2) = \sum_{t_1 \in k, t_2 \in k^\times} \widetilde{\varphi_f}(t_1, t_2),$$

where $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$.

Proof. According to Remark 3.2.34, we have

$$\frac{\mathcal{H}_{\mathrm{Std},\nu}(f_\nu)}{(\mathrm{d}t)^{\frac{1}{2}}}(t) = |t_1|^{-\frac{1}{2}} |t_2|^{\frac{1}{2}} \int_{k_\nu} \widetilde{\varphi_\nu}(t_1, t_2) \psi\left(\frac{x}{t_1}\right) \mathrm{d}x,$$

hence

$$\sum_{t \in \mathrm{T}(k)} \frac{\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)}{(\mathrm{d}t)^{\frac{1}{2}}}(t) = \sum_{(t_1,t_2) \in k^\times \times k^\times} \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\varphi_f}(\cdot, t_2)) \left(-\frac{1}{t_1}\right)$$

since $|t_1| = |t_2| = 1$ for $t_1, t_2 \in k^\times$. Thanks to Lemma 3.2.38, we have

$$\mathcal{O}^\vee_{u,1,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)) = \sum_{t_2 \in k^\times} |t_2| \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\varphi_f}(\cdot, t_2))(0) = \sum_{t_2 \in k^\times} \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\varphi_f}(\cdot, t_2))(0).$$

Combining them, we have

$$\begin{aligned}
&\sum_{t\in \mathrm{T}(k)} \frac{\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t) + \sum_{t_2\in k^\times} \mathcal{O}^\vee_{u,1,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f))(t_2)\\
&= \sum_{(t_1,t_2)\in k^\times\times k^\times} \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\varphi_f}(\cdot,t_2))\left(-\frac{1}{t_1}\right) + \sum_{t_2\in k^\times} \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\psi_f}(\cdot,t_2))(0)\\
&= \sum_{t_2\in k^\times}\sum_{t_1\in k} \mathbb{F}_{\psi,1,\mathbb{A}}(\widetilde{\varphi_f}(\cdot,t_2))(t_1).
\end{aligned}$$

Then, for each fixed $t_2 \in k^\times$, since $\widetilde{\varphi_f}$ is a function of Schwartz type according to Proposition 3.2.24, we can apply the classical Poisson summation formula to obtain the above is

$$\sum_{t_1\in k, t_2\in k^\times} \widetilde{\varphi_f}(t_1,t_2).$$

□

Lemma 3.3.8. *We have*

$$\sum_{t_1\in k,t_2\in k^\times} \widetilde{\varphi_f}(t_1,t_2) - \sum_{t_2\in k^\times} \mathcal{O}_{u,2,\mathbb{A}}(f)(t_2) + \sum_{t_1\in k^\times} \mathcal{O}^\vee_{u,2,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f))(t_1)$$
$$= \sum_{t\in\mathrm{T}(k)} \frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t) + \sum_{t_1\in k^\times} \mathcal{O}_{u,1,\mathbb{A}}(f)(t_1).$$

Proof. According to Definition 3.2.39, we have

$$\sum_{t_2\in k^\times} \mathcal{O}_{u,2,\mathbb{A}}(f) = \sum_{t_2\in k^\times} \widetilde{\varphi_f}(0,t_2).$$

Write $f_\nu = |\det t|_\nu \delta^{\frac{1}{2}}(t)\mathcal{O}_t(\Phi_\nu)$ and $\mathcal{O}_t(\Phi) = \otimes'_\nu \mathcal{O}_t(\Phi_\nu)$, then according to Lemma 3.2.43, we have

$$\mathcal{O}^\vee_{u,2,\mathbb{A}}(\mathcal{H}_{\mathrm{Std},\mathbb{A}}(f)) = \sum_{z\in k^\times} |z|^3 \mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0) = \sum_{z\in k^\times} \mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0).$$

Therefore, the left-hand-side of the equality in this lemma is

$$\begin{aligned}
&\sum_{t_1\in k,t_2\in k^\times} \widetilde{\varphi_f}(t_1,t_2) - \sum_{t_2\in k^\times} \widetilde{\varphi_f}(0,t_2) + \sum_{z\in k^\times} \mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0)\\
&= \sum_{t_1\in k^\times,t_2\in k^\times} \widetilde{\varphi_f}(t_1,t_2) + \sum_{z\in k^\times} \mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0).
\end{aligned}$$

Use Lemma 3.2.23, we have the above is

$$\begin{aligned}
&\sum_{t_1\in k^\times, t_2\in k^\times} |t_1|\psi\left(-\frac{t_2}{t_1}\right)\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(t_1,\cdot)}(\Phi))\left(-\frac{1}{t_2}\right)+\sum_{z\in k^\times}\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0)\\
&=\sum_{t_1\in k^\times, t_2\in k^\times}\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(t_1,\cdot)}(\Phi))\left(-\frac{1}{t_2}\right)+\sum_{z\in k^\times}\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(-z,\cdot)}(\Phi))(0)\\
&=\sum_{t_1\in k^\times}\sum_{t_2\in k}\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(t_1,\cdot)}(\Phi))(t_2).
\end{aligned}$$

According to Proposition 3.2.21, for each $t_1 \in k^\times$, $\mathcal{O}_{(t_1,\cdot)}(\Phi)$ is a function of Schwartz type, we can apply the Poisson summation formula to it. Therefore we obtain

$$\begin{aligned}
\sum_{t_1\in k^\times}\sum_{t_2\in k}\mathbb{F}_{\psi,2,\mathbb{A}}(\mathcal{O}_{(t_1,\cdot)}(\Phi))(t_2) &= \sum_{t_1\in k^\times}\sum_{t_2\in k}\mathcal{O}_{(t_1,t_2)}(\Phi)\\
&=\sum_{t_1\in k^\times, t_2\in k^\times}\mathcal{O}_{(t_1,t_2)}(\Phi)+\sum_{t_1\in k^\times}\mathcal{O}_{(t_1,0)}(\Phi)\\
&=\sum_{t\in \mathrm{T}(k)}\frac{f}{(\,\mathrm{d}t)^{\frac{1}{2}}}(t)+\sum_{t_1\in k^\times}\mathcal{O}_{u,1,\mathbb{A}}(f)(t_1).
\end{aligned}$$

□

Combining these two lemmas, we get Theorem 3.3.6.

3.4 Kuznetsov Trace Formula for Standard L-functions of GL_2

In this section, we will first establish a Kuznetsov-type trace formula with test functions being the Schwartz functions on the space of 2×2 matrices. This is one example of a trace formula with non-standard test functions. The geometric side will be related to Theorem 3.3.6, which allows us to apply the Poisson summation formula on the geometric side. The spectrum side has already been studied in [Wu22].

Let Φ be a Schwartz function on $\mathrm{Mat}_2(\mathbb{A})$. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have $g \mapsto \Phi(g)|\det g|^s \cdot |\det g| \in \mathcal{L}^1(\mathrm{GL}_2)$. For any unitary idele class character ω, consider the right regular action R_ω on $\mathcal{L}^2(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}),\omega)$. By calculating the Whittaker period of the kernel function of $\mathrm{R}(\Phi(\cdot)|\det\cdot|^{s+1})$, we will prove the following

Theorem 3.4.1. *For $\Phi \in \mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have*

$$
\begin{aligned}
&\frac{1}{(\,\mathrm{d}x)^{\frac{1}{2}}}\int_{[\mathrm{Z}]} \mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z \\
&= \sum_{\pi \text{ cuspidal}, \omega_\pi = \omega} \sum_{\varphi_1 . \varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^\vee)\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2}^{\psi^{-1}}(\mathrm{I}_2) \\
&+ \sum_{\chi \in \widehat{\mathbb{R}_+ k^\times \backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1, e_2 \in \mathcal{B}(\chi, \omega\chi^{-1})} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta_{i\tau}(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(i\tau, e_1)(1)\mathbb{W}^{\psi}(-i\tau, e_2^\vee)(1)\frac{\mathrm{d}\tau}{4\pi} \\
&+ \int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}'(n_1, n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,
\end{aligned}
$$

with the last term being the Whittaker function of the Eisenstein series. For the undefined terms, see the next two sections.

3.4.1 The Spectral Side of the Trace Formula

First, let us set up notations for the spectral side. We fix a section of the adelic norm map $|\cdot| : \mathbb{A}^\times \to \mathbb{R}_+$ and identify $\mathbb{R}_+$ as a subgroup of $\mathbb{A}^\times$. Let ω be a unitary idele class character that is also trivial on $\mathbb{R}_+$. Denote $\mathcal{L}^2(\mathrm{GL}_2, \omega)$ the space of functions $\varphi : \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ such that

(1) $\varphi(gz) = \omega(z)\varphi(g)$, $\forall g \in \mathrm{GL}_2(\mathbb{A})$ and $z \in \mathrm{Z}(\mathbb{A})$,

(2) $\displaystyle\int_{Z(\mathbb{A})\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})} |\varphi(g)|^2\,\mathrm{d}g < \infty.$

Then $\mathrm{GL}_2(\mathbb{A})$ acts on it via the right regular action, and we denote this action by

$$(\mathrm{R}_\omega, \mathcal{L}^2(\mathrm{GL}_2, \omega)).$$

Let (π, V_π) be a cuspidal automorphic representation in $(\mathrm{R}_\omega, \mathcal{L}^2(\mathrm{GL}_2, \omega))$, we will use V_π^∞ to denote the subspace of smooth vectors. For $\varphi \in V_\pi^\infty$, we write

$$\mathbb{W}_\varphi^\psi(g) := \int_{[\mathrm{N}]} \varphi(ng)\psi^{-1}(n)\,\mathrm{d}n$$

for the Whittaker function of φ.

Similarly let χ be a unitary character of $\mathbb{R}_+k^\times\backslash\mathbb{A}^\times$, we can associate the principal series $\pi_s := \pi(\chi|\cdot|^s, \omega\chi^{-1}|\cdot|^{-s})$ for all $s \in \mathbb{C}$, whose underlying Hilbert space is $V_{\chi,\omega\chi^{-1}}$ consisting all the functions $f : \mathrm{K} \to \mathbb{C}$ satisfying

(1) $f\left(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} g\right) = \chi(t_1)\omega\chi^{-1}(t_2)f(g)$,

(2) $\int_{\mathrm{K}} |f(k)|^2 \, \mathrm{d}k < \infty$,

where K is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$, which is the tensor product of K_ν, and

$$\mathrm{K}_\nu = \begin{cases} \mathrm{SO}_2(\mathbb{R}) & \text{if} \quad k_\nu \cong \mathbb{R} \\ \mathrm{SU}_2(\mathbb{C}) & \text{if} \quad k_\nu \cong \mathbb{C} \\ \mathrm{GL}_2(\mathfrak{o}_\nu) & \text{if} \quad \nu < \infty \end{cases}.$$

This space is common for all $s \in \mathbb{C}$ and the subspace of smooth vectors, although the actions π_s are different. In particular, π_s is unitary if and only if $s \in i\mathbb{R}$. To any smooth vector $e \in V^\infty_{\chi,\omega\chi^{-1}}$, we associate a flat section $e(s)$ in π_s, from which we construct an Eisenstein series

$$\mathrm{E}(s,e)(g) := \sum_{\gamma \in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} e(s)(\gamma g),$$

which converges for $\mathrm{Re}(s) \gg 1$ and admits a meromorphic continuation to $s \in \mathbb{C}$. We write

$$\mathbb{W}^\psi(s,e)(g) := \int_{[\mathrm{N}]} \mathrm{E}(s,e)(ng)\psi^{-1}(n)\,\mathrm{d}n = \int_{\mathrm{N}(\mathbb{A})} e(wng)\psi^{-1}(n)\,\mathrm{d}n$$

for the Whittaker function of $\mathrm{E}(s,e)$.

For $V = V_\pi$ or $V_{\chi,\omega\chi^{-1}}$, the underlying inner product identifies V with its dual $V^\vee$. Define $\varphi^\vee := \frac{\overline{\varphi}}{\|\varphi\|^2} \in V^\vee$ for $\varphi \neq 0$. The matrix coefficient $\beta(\varphi_2, \varphi_1^\vee)$ and $\beta_s(e_2, e_1^\vee)$ for a pair of non-zero vectors is defined to be

$$\beta(\varphi_2, \varphi_1^\vee)(g) := \frac{\langle \pi(g)\varphi_2, \varphi_1 \rangle}{\|\varphi_1\|^2} \text{ and } \beta_s(e_2, e_1^\vee)(g) := \frac{\langle \pi_s(g)e_2, e_1 \rangle}{\|e_1\|^2}, \ \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

For a Schwartz function Φ on $\mathrm{Mat}_2(\mathbb{A})$, we have the Godement-Jacquet zeta integrals for $\beta \in \{\beta(\varphi_2, \varphi_1^\vee), \beta_s(e_2, e_1^\vee)\}$

$$\mathcal{Z}(s', \Phi, \beta) := \int_{\mathrm{GL}_2(\mathbb{A})} \Phi(g)\beta(g)|\det g|^{s' + \frac{1}{2}}\,\mathrm{d}g,$$

which converges for $\mathrm{Re}(s) \gg 1$.

Let Φ be a Schwartz function on $\mathrm{Mat}_2(\mathbb{A})$. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have $g \mapsto \Phi(g)|\det g|^s \cdot |\det g| \in \mathcal{L}^1(\mathrm{GL}_2)$ and it acts on $(\mathrm{R}_\omega, \mathcal{L}^2(\mathrm{GL}_2, \omega))$. For any

$$\varphi \in \mathcal{L}^2(\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}), \omega),$$

we have for $x \in \mathrm{GL}_2(\mathbb{A})$,

$$\begin{aligned}
&\mathrm{R}_\omega(\Phi(\cdot)|\det \cdot|^{s+1})\varphi(x) = \int_{\mathrm{GL}_2(\mathbb{A})} \Phi(y)|\det y|^{s+1}\varphi(xy)\,\mathrm{d}y \\
&= \int_{\mathrm{GL}_2(\mathbb{A})} \Phi(x^{-1}y)|\det x^{-1}y|^{s+1}\varphi(y)\,\mathrm{d}y \\
&= \int_{[\mathrm{GL}_2]} \sum_{\gamma \in \mathrm{GL}_2(k)} \Phi(x^{-1}\gamma y)|\det x^{-1}y|^{s+1}\varphi(y)\,\mathrm{d}y \\
&= \int_{\mathrm{GL}_2(k)\mathrm{Z}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \int_{[\mathrm{Z}]} \sum_{\gamma \in \mathrm{GL}_2(k)} \Phi(x^{-1}\gamma yz)|z|^{2s+2}\omega(z)\,\mathrm{d}^\times z|\det x^{-1}y|^{s+1}\varphi(y)\,\mathrm{d}y.
\end{aligned}$$

We denote the corresponding kernel function

$$\mathrm{K}(x,y) := \int_{[\mathrm{Z}]} \sum_{\gamma \in \mathrm{GL}_2(k)} \Phi(x^{-1}\gamma yz)|z|^{2s+2}\omega(z)\,\mathrm{d}^\times z \cdot |\det x^{-1}y|^{s+1}.$$

According to [Wu22, Theorem 2.10], we have the $\mathcal{L}^2$-spectrum decomposition

Theorem 3.4.2. *When* $\mathrm{Re}(s) > 1$*, the Fourier inversion of* $\mathrm{K}(x,y)$ *with respect to* y *in* $\mathcal{L}^2(\mathrm{GL}_2, \omega^{-1})$ *converges normally for* $(x,y) \in [\mathrm{GL}_2] \times [\mathrm{GL}_2]$*, and takes the form*

$$\begin{aligned}
\mathrm{K}(x,y) = &\sum_{\pi \text{ cuspidal}, \omega_\pi = \omega} \sum_{\varphi_1.\varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^\vee)\right)\varphi_1(x)\varphi_2^\vee(y) \\
&+ \sum_{\chi \in \mathbb{R}_+\widehat{k^\times\backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1,e_2 \in \mathcal{B}(\chi, \omega\chi^{-1})} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta_{i\tau}(e_2, e_1^\vee)\right)\mathrm{E}(i\tau, e_1)(x)\mathrm{E}(-i\tau, e_2^\vee)(y)\frac{\mathrm{d}\tau}{4\pi} \\
&+ \frac{1}{\mathrm{vol}[\mathrm{PGL}_2]} \sum_{\eta \in \widehat{k^\times\backslash \mathbb{A}^\times}, \eta^2 = \omega} \left(\Phi(g)\eta(\det g)|\det g|^{s+1}\,\mathrm{d}g\right)\eta(\det x)\overline{\eta(\det y)},
\end{aligned}$$

where $\mathcal{B}(\pi)$ *is an orthogonal basis of* K*-isotypic vectors in* π*,* $\mathcal{B}(\chi, \omega\chi^{-1})$ *is an orthogonal basis of* K*-isotypic vectors in* $V_{\chi,\omega\chi^{-1}}$*.*

Integrating the above formula against the prescribed character over $[\mathrm{N}] \times [\mathrm{N}]$, we obtain

$$\begin{aligned}
&\int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \\
&= \sum_{\pi \text{ cuspidal},\omega_\pi=\omega}\ \sum_{\varphi_1.\varphi_2\in\mathcal{B}(\pi)} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta(\varphi_2,\varphi_1^\vee)\right)\mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2) \\
&+ \sum_{\chi\in\mathbb{R}_+\widehat{k^\times\backslash\mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_{i\tau}(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(i\tau,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}.
\end{aligned}$$

3.4.2 The Geometric Side of the Trace Formula

In this section, we will calculate the integral of $\mathrm{K}(x,y)$ with the prescribed character over $[\mathrm{N}]\times[\mathrm{N}]$ from the geometric aspect.

When $\mathrm{Re}(s)>1$, we have

$$\begin{aligned}
&\int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \\
&= \int_{[\mathrm{N}]\times[\mathrm{N}]}\int_{[\mathrm{Z}]}\sum_{\gamma\in\mathrm{GL}_2(k)} \Phi(n_1^{-1}\gamma n_2 z)|z|^{2s+2}\omega(z)\,\mathrm{d}^\times z\cdot\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \\
&= \int_{[\mathrm{Z}]}\int_{[\mathrm{N}]\times[\mathrm{N}]} \Phi(n_1\gamma n_2 z)\psi^{-1}(n_1n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\cdot\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z.
\end{aligned}$$

In addition to the kernel function $\mathrm{K}(x,y)$, we can also consider the following function given by a similar summation over rank 1 rational matrices when $\mathrm{Re}(s)\gg 1$:

$$\mathrm{K}'(x,y):=\int_{[\mathrm{Z}]}\sum_{\gamma\in\mathrm{Mat}_2(k),\mathrm{rank}\gamma=1}\Phi(x^{-1}\gamma yz)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\cdot|\det x^{-1}y|^{s+1}.$$

According to [Wu22, Section 2.2], when $\mathrm{Re}(s)\gg 1$, denote

$$\begin{aligned}
\mathrm{R}(x,y) &:= \int_{[\mathrm{Z}]}\sum_{\gamma\in k^\times}\Phi\left(x^{-1}\begin{pmatrix}0&\gamma\\0&0\end{pmatrix}yz\right)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\cdot|\det x^{-1}y|^{s+1} \\
&= \int_{\mathbb{A}^\times}\Phi\left(x^{-1}\begin{pmatrix}0&z\\0&0\end{pmatrix}y\right)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\cdot|\det x^{-1}y|^{s+1}.
\end{aligned}$$

It is easy to see that for fixed x, $y\mapsto\mathrm{R}(x,y)$ belongs to the induced model of the principle series representation $\pi(|\cdot|^{s+\frac{1}{2}},\omega^{-1}|\cdot|^{-s-\frac{1}{2}})$, and for fixed y, $x\mapsto\mathrm{R}(x,y)$ is in

$\pi(\omega|\cdot|^{s+\frac{1}{2}}, |\cdot|^{-s-\frac{1}{2}})$. Note that the action of $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ on the rank 1 rational matrices admits a single orbit and the stabilizer group of the line consisting of the elements

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}, \gamma \in k^\times$$

is $\mathrm{B}(k) \times \mathrm{B}(k)$, therefore,

$$\mathrm{K}'(x,y) = \sum_{\gamma_1 \in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \sum_{\gamma_2 \in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \mathrm{R}(\gamma_1 x, \gamma_2 y)$$

is an Eisenstein series in each variable.

For notational simplicity, let us still write KTF for the composition of KTF and the twisted push-forward from $\mathcal{D}(\mathrm{Mat}_2(\mathbb{A}))$ to $\mathcal{D}_{L\left(\mathrm{Std},\frac{1}{2}\right)}(\mathrm{N}(\mathbb{A}),\psi\backslash \mathrm{G}/\mathrm{N}(\mathbb{A}),\psi)$. Then, the geometric side can be written as

$$\int_{[\mathrm{Z}]} \mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z - \int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}'(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,\ \mathrm{Re}(s) \gg 1,$$

and Theorem 3.4.1 holds.

3.4.3 Analytic Continuation

In this section, we will prove the analytic continuation of each term in Theorem 3.4.1. It will be independent of the Godement-Jacquet theory and the Poisson summation formula on $\mathrm{Mat}_2(\mathbb{A})$. For simplicity, denote

$$\mathcal{I}(s) := \int_{[\mathrm{Z}]} \mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z,\ \mathrm{Re}(s) > 1,$$

$$\mathcal{I}_{\mathrm{b}}(s) := \int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}'(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2,\ \mathrm{Re}(s) \gg 1,$$

$$\mathcal{I}_{\mathrm{cusp}}(s) := \sum_{\pi \text{ cuspidal}, \omega_\pi = \omega} \sum_{\varphi_1 . \varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^\vee)\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2),$$

and also when $\mathrm{Re}(s) > 1$:

$$\mathcal{I}_{\mathrm{Eis}}(s)$$
$$:= \sum_{\chi \in \widehat{\mathbb{R}_+ k^\times \backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1, e_2 \in \mathcal{B}(\chi, \omega\chi^{-1})} \mathcal{Z}\left(s + \frac{1}{2}, \Phi, \beta_{i\tau}(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(i\tau, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}.$$

We may choose Φ to be K-finite, hence the summation of χ, e_1, e_2 in $\mathcal{I}_{\mathrm{Eis}}$ is finite and we denote

$$\mathcal{I}_{\chi,e_1,e_2}(s) := \int_{-\infty}^{\infty} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_{i\tau}(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(i\tau, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}.$$

Let us first study the convergence of $\mathcal{I}(s)$. According to the calculation in Section 3.3.1, write

$$\mathbb{O}_1 = \bigsqcup_{\beta\in k^\times,\gamma\in k} \mathrm{N}(k)\begin{pmatrix}0 & \gamma\\ \beta & 0\end{pmatrix}\mathrm{N}(k),\ \mathbb{O}_2 = \bigsqcup_{\alpha\in k^\times} \mathrm{N}(k)\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\mathrm{N}(k),$$

and

$$\mathrm{K}_i(z) := \int_{[\mathrm{N}]\times[\mathrm{N}]} \sum_{\gamma\in\mathbb{O}_i} \Phi(n_1\gamma n_2 z)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_2\,\mathrm{d}n_2,$$

then we have

$$\mathcal{I}(s) = \int_{[\mathrm{Z}]} (\mathrm{K}_1(z)+\mathrm{K}_2(z))\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z.$$

For the first term, we have

$$\begin{aligned}\mathrm{K}_1(z) &= \int_{\mathbb{A}^2} \sum_{\beta\in k^\times,\gamma\in k} \Phi\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \gamma\\ \beta & 0\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix} z\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\\ &= \int_{\mathbb{A}^2} \sum_{\beta\in k^\times,\gamma\in k} \Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z+\beta n_1 n_2 z\\ \beta z & \beta n_2 z\end{pmatrix}\right)\psi^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.\end{aligned}$$

Applying the Poisson summation formula, we have

$$\sum_{\gamma\in k}\Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z+\beta n_1 n_2 z\\ \beta z & \beta n_2 z\end{pmatrix}\right) = |z|^{-1}\sum_{\gamma\in k}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z^{-1}\\ \beta z & \beta n_2 z\end{pmatrix}\right)\psi(\beta n_1 n_2\gamma z^{-1}).$$

Integral over $n_2\in\mathbb{A}$, we have

$$\begin{aligned}&\int_{n_2\in\mathbb{A}}\sum_{\gamma\in k}\Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z+\beta n_1 n_2 z\\ \beta z & \beta n_2 z\end{pmatrix}\right)\psi^{-1}(n_2)\,\mathrm{d}n_2\\ &= |z|^{-1}\int_{n_2\in\mathbb{A}}\sum_{\gamma\in k}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z^{-1}\\ \beta z & \beta n_2 z\end{pmatrix}\right)\psi^{-1}(n_2-\beta n_1 n_2\gamma z^{-1})\,\mathrm{d}n_2\\ &= |z|^{-2}\sum_{\gamma\in k}\mathbb{F}_{\psi,4}\mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix}\beta n_1 z & \gamma z^{-1}\\ \beta z & \beta^{-1}z^{-1}-n_1\gamma z^{-2}\end{pmatrix}\right).\end{aligned}$$

We need the following lemma:

Lemma 3.4.3. *Let $\Phi \in \mathcal{F}(\mathbb{A})$. Then we have for any $N > 1$,*

$$\sum_{\alpha \in k^\times} |\Phi(\alpha z)| \ll_N \min\{|z|^{-1}, |z|^{-N}\}.$$

Proof. This elementary estimation appeared in [GJ72] without proof. For a detailed proof, see [Wu19]. $\square$

According to the above lemma, we have

$$\mathrm{K}_1(z) \ll_{N_1} |z|^{-3} \min\{|z|^{-1}, |z|^{-N_1}\}(1 + \min |z|, |z|^{N_2}).$$

As for $\mathrm{K}_2(z)$, we have

$$\begin{aligned} \mathrm{K}_2(z) &= \int_{\mathbb{A}} \sum_{\alpha \in k^\times} \Phi\left(\begin{pmatrix} \alpha z & \alpha z n \\ 0 & \alpha z \end{pmatrix}\right) \psi^{-1}(n)\, \mathrm{d}n \\ &= |z|^{-1} \sum_{\alpha \in k^\times} \mathbb{F}_{\psi,2}\Phi\left(\begin{pmatrix} \alpha z & \alpha^{-1} z^{-1} \\ 0 & \alpha z \end{pmatrix}\right) \\ &\ll_{N_1,N_2} |z|^{-1} \min\{|z|^{-1}, |z|^{-N_1}\} \min\{|z|, |z|^{N_2}\}. \end{aligned}$$

Combining all these, we have

Proposition 3.4.4.

$$\int_{z \in [\mathrm{Z}], |z| \geq 1} \mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\mathrm{d}^+ x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\, \mathrm{d}^\times z$$

is convergent for all $s \in \mathbb{C}$ and defines an entire function.

When $\mathrm{Re}(s) > 1$, by applying Theorem 3.3.6, we have

$$\mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\mathrm{d}x)^{\frac{1}{2}})) = \mathrm{KTF}(\widehat{\mathrm{R}_z(\Phi)}(x)(\mathrm{d}x)^{\frac{1}{2}}),$$

and note that

$$\widehat{\mathrm{R}_z(\Phi)}(x) = |z|^{-4}\widehat{\Phi}(z^{-1}x), \ \forall x \in \mathrm{Mat}_2(\mathbb{A}), z \in Z(\mathbb{A}).$$

Hence we have

$$\widehat{\mathrm{R}_z(\Phi)}(x)(\mathrm{d}x)^{\frac{1}{2}} = \mathrm{R}_z(\widehat{\Phi}(x)(\mathrm{d}^+ x)^{\frac{1}{2}})|z|^{-2}.$$

Therefore,

$$\begin{aligned}&\int_{|z|<1}\mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z\\&=\int_{|z|<1}\mathrm{KTF}(\mathrm{R}_z(\widehat{\Phi}(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z,\end{aligned}$$

which is absolutely convergent for all $s\in\mathbb{C}$ and defines an entire function according to the above Proposition and the $\mathrm{GL}_2\times\mathrm{GL}_2$ action on $\mathrm{Mat}_2^\vee$ that we defined in Section 3.2.1. Therefore we have

Proposition 3.4.5. *$\mathcal{I}(s)$ admits an analytic continuation to an entire function on $s\in\mathbb{C}$.*

Next we will prove the analytic continuation of $\mathcal{I}_{\mathrm{Eis}}$, for which it suffices to show the analytic continuation of $\mathcal{I}_{\chi,e_1,e_2}(s)$ for each triple (χ,e_1,e_2). Write z for the coordinate of the inner integral of τ, i.e.,

$$\begin{aligned}\mathcal{I}_{\chi,e_1,e_2}(s)&=\int_{-\infty}^{\infty}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_{i\tau}(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(i\tau,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}\\&=\int_{\mathrm{Re}(z)=0}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_{z}(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}.\end{aligned}$$

According to [Wu22, proof of Theorem 2.10], we have

$$\begin{aligned}&\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\\&=\int_{(\mathbb{A}^\times)^2}\mathbb{F}_{\psi,2}({}_{e_1^\vee}\Phi_{e_2})\left(\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}\right)|t_1|^{s+\frac{1}{2}+z}\chi(t_1)\omega\chi^{-1}(t_2)|t_2|^{s+\frac{1}{2}-z}\,\mathrm{d}^\times t_1\,\mathrm{d}^\times t_2,\end{aligned}$$

where

$${}_{e_1^\vee}\Phi_{e_2}(g):=\int_{\mathrm{K}^2}e_1^\vee(k_1)\Phi(k_1^{-1}gk_2)e_2(k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2.$$

Then according to [Tat67], we know $\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)$ is a meromorphic function by itself. Then, we need to consider the case that the zeta integrals have poles on the line $\mathrm{Re}(z)=0$, which could occur only when $\chi=1$ and $\chi=\omega$.

For any $0<\sigma<\frac{1}{2}$, and consider the case when $\frac{1}{2}<\mathrm{Re}(s)<\frac{1}{2}+\sigma$. According to Stirling's formula, we know $\mathcal{Z}\left(s+\dfrac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)$ is of rapid decay when $\mathrm{Re}(z)\to\infty$

for $0<\mathrm{Re}(z)<\sigma$. According to [JZ87, Lemma 1, Page8], the Whittaker functions at value I_2 is uniformly bounded on vertical strips, we can move the z-integral from $\mathrm{Re}(z)=0$ to $\mathrm{Re}(z)=\sigma$ and pick up the poles, we obtain

$$\begin{aligned}
&\int_{\mathrm{Re}(z)=0}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&=\int_{\mathrm{Re}(z)=\sigma}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&-\frac{1}{2}\sum_{0<\mathrm{Re}(z)<\sigma,s+\frac{1}{2}\pm z=0,1}\mathrm{Res}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2).
\end{aligned}$$

Note that the only possible residue in this region occurs when $s+\frac{1}{2}-z=1$, which implies $0<\mathrm{Re}(z)<\sigma$ automatically, and hence the residue part is just

$$-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\cdot\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2).$$

Since there is no pole on the line $\mathrm{Re}(z)=\sigma$ when $\frac{1}{2}-\sigma<\mathrm{Re}(s)<\frac{1}{2}+\sigma$, and the integral is absolutely convergent. So we can extend $\mathcal{I}_{\chi,e_1,e_2}(s)$ to $\mathrm{Re}(s)>0$.

When $\frac{1}{2}-\sigma<\mathrm{Re}(s)<\frac{1}{2}$, let us then move the z-integral to $\mathrm{Re}(z)=0$ again and pick up the poles, we have

$$\begin{aligned}
&\int_{\mathrm{Re}(z)=\sigma}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&-\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2)\\
&=\int_{\mathrm{Re}(z)=0}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2).
\end{aligned}$$

Note that this can be analytically extended to $-\frac{1}{2}<\mathrm{Re}(s)<\frac{1}{2}$ since there is pole on the line $\mathrm{Re}(z)=0$ in this region.

Next consider the case that $\mathrm{Re}(s)<-\frac{1}{2}+\sigma$, let us move the integral from $\mathrm{Re}(z)=0$

to $\mathrm{Re}(z)=\sigma$ again, we have

$$\begin{aligned}
&\int_{\mathrm{Re}(z)=0} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2)\\
&=\int_{\mathrm{Re}(z)=\sigma} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&-\frac{1}{2}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2),
\end{aligned}$$

which can be analytically continued to $-\sigma-\frac{1}{2}<\mathrm{Re}(s)<-\frac{1}{2}+\sigma$. When $-\sigma-\frac{1}{2}<\mathrm{Re}(s)<-\frac{1}{2}$, we could move the integral to $\mathrm{Re}(z)=0$ again, and we have finally

$$\begin{aligned}
&\int_{\mathrm{Re}(z)=\sigma} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&-\frac{1}{2}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2)\\
&=\int_{\mathrm{Re}(z)=0} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&+\frac{1}{2}\mathrm{Res}_{z=-s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(-s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s+\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2).
\end{aligned}$$

And this can be extended to all $\mathrm{Re}(s) < -\frac{1}{2}$. To summarize, we have

Proposition 3.4.6. *$\mathcal{I}_{\mathrm{Eis}}(s)$ can be analytically extended to all $s \in \mathbb{C}$. Moreover, for each χ, e_2, e_1, the analytic continuation of*

$$\mathcal{I}_{\chi,e_1,e_2}(s) = \int_{\mathrm{Re}(z)=0} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(z, e_1)(1)\mathbb{W}^{\psi}(-z, e_2^\vee)(1)\frac{\mathrm{d}z}{4\pi i},\ \mathrm{Re}(s) > \frac{1}{2}$$

has the following expressions:

- $0 < \mathrm{Re}(s) \leq \dfrac{1}{2}$:

$$\begin{aligned}\mathcal{I}_{\chi,e_1,e_2} &= \int_{\mathrm{Re}(z)=\sigma} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(z, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i} \\ &- \frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2}, e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s, e_2^\vee\right)(\mathrm{I}_2),\end{aligned}$$

  where $0 < \sigma < \dfrac{1}{2}$ is such that $\dfrac{1}{2} - \sigma < \mathrm{Re}(s)$.

- $-\dfrac{1}{2} < \mathrm{Re}(s) \leq 0$:

$$\begin{aligned}\mathcal{I}_{\chi,e_1,e_2} &= \int_{\mathrm{Re}(z)=0} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(z, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i} \\ &+ \frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s, e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2}, e_2^\vee\right)(\mathrm{I}_2) \\ &- \frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2}, e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s, e_2^\vee\right)(\mathrm{I}_2).\end{aligned}$$

- $-1 < \mathrm{Re}(s) \leq -\dfrac{1}{2}$:

$$\begin{aligned}\mathcal{I}_{\chi,e_1,e_2} &= \int_{\mathrm{Re}(z)=\sigma} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(z, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i} \\ &- \frac{1}{2}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2}, e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2}, e_2^\vee\right)(\mathrm{I}_2) \\ &+ \frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(s-\frac{1}{2}, e_2^\vee\right)(\mathrm{I}_2) \\ &- \frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2}, e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s, e_2^\vee\right)(\mathrm{I}_2),\end{aligned}$$

  where $0 < \sigma < \dfrac{1}{2}$ is such that $-\dfrac{1}{2} - \sigma < \mathrm{Re}(s)$.

- $\mathrm{Re}(s) \leq -1$:

$$
\begin{aligned}
\mathcal{I}_{\chi,e_1,e_2}(s) &= \int_{\mathrm{Re}(z)=0} \mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}(z,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-z,e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}z}{4\pi i}\\
&+\frac{1}{2}\mathrm{Res}_{z=-s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(-s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s+\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&+\frac{1}{2}\mathrm{Res}_{z=\frac{1}{2}-s}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(\frac{1}{2}-s,e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\
&-\frac{1}{2}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(\frac{1}{2}-s,e_2^\vee\right)(\mathrm{I}_2).
\end{aligned}
$$

We will conclude this section with a more careful study of the residues occurring in the above analytic continuation of $\mathcal{I}_{\chi,e_1,e_2}(s)$. Let us start with the following definitions from [Wu22, Section 4]:

Definition 3.4.7. *For any Schwartz function $\Phi \in \mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))$, and a unitary character χ of $\mathbb{R}_+k^\times\backslash\mathbb{A}^\times$, we define for $g_1, g_2 \in \mathrm{GL}_2(\mathbb{A})$ and $\mathrm{Re}(s) \gg 1$,*

$$
f_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},*) := \int_{\mathbb{A}^\times}\mathbb{F}_{\psi,2}\mathrm{L}(g_1)\mathrm{R}(g_2)\Phi\left(\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix}\right)\chi(t)|t|^{2s+1}\,\mathrm{d}^\times t\cdot|\det g_1^{-1}g_2|^{s+1},
$$

and

$$
f_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},\star) := \int_{\mathbb{A}^\times}\mathbb{F}_{\psi,2}\mathrm{L}(g_1)\mathrm{R}(g_2)\Phi\left(\begin{pmatrix} 0 & 0\\ 0 & t\end{pmatrix}\right)\chi(t)|t|^{2s+1}\,\mathrm{d}^\times t\cdot|\det g_1^{-1}g_2|^{s+1}.
$$

Both of them can be analytically extended to $s \in \mathbb{C}$. Note that $f_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},*)$ defines a vector in the tensor product $\pi(\chi|\cdot|^{s+\frac{1}{2}},|\cdot|^{-s-\frac{1}{2}})\otimes\pi(\chi^{-1}|\cdot|^{-s-\frac{1}{2}},|\cdot|^{s+\frac{1}{2}})$. We denote by $(\mathcal{M}\times f)_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},*)$ the image under the intertwining operator with respect to the first variable. Similarly, $f_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},\star) \in \pi(|\cdot|^{-s-\frac{1}{2}},\chi|\cdot|^{s+\frac{1}{2}})\otimes\pi(|\cdot|^{s+\frac{1}{2}},\chi^{-1}|\cdot|^{-s-\frac{1}{2}})$, and we denote $(f\times\mathcal{M})_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},\star)$ its image under the intertwining operator with respect to the second variable.

Lemma 3.4.8. *We have*

$$
(\mathcal{M}\times f)_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},*) = (f\times\mathcal{M})_\Phi(g_1,g_2;\chi|\cdot|^{2s+\frac{3}{2}},\star).
$$

Proof. It suffices to prove the equation for $g_1 = g_2 = \mathrm{I}_2$ since the general case follows from this special case by taking $\mathrm{L}(g_1)\mathrm{R}(g_2)\Phi$ instead of Φ. By definition, when $\mathrm{Re}(s) \gg$

1, we have

$$
\begin{aligned}
&(\mathcal{M}\times f)_{\Phi}(\mathrm{I}_2,\mathrm{I}_2;\chi|\cdot|^{2s+\frac{3}{2}},*)\\
&=\int_{\mathbb{A}}\int_{\mathbb{A}^\times}\int_{\mathbb{A}}\Phi\left(\begin{pmatrix}1 & -n\\ 0 & 1\end{pmatrix}w^{-1}\begin{pmatrix}t & x\\ 0 & 0\end{pmatrix}\right)\mathrm{d}x\chi(t)|t|^{2s+1}\,\mathrm{d}^\times t\,\mathrm{d}n\\
&=\int_{\mathbb{A}}\int_{\mathbb{A}^\times}\int_{\mathbb{A}}\Phi\left(\begin{pmatrix}0 & -nt\\ 0 & t\end{pmatrix}w\begin{pmatrix}1 & xt^{-1}\\ 0 & 1\end{pmatrix}\right)\mathrm{d}x\chi(t)|t|^{2s+1}\,\mathrm{d}^\times t\,\mathrm{d}n\\
&=(f\times\mathcal{M})_{\Phi}(\mathrm{I}_2,\mathrm{I}_2;\chi|\cdot|^{2s+\frac{3}{2}},\star).
\end{aligned}
$$

□

Proposition 3.4.9. *When* $\mathrm{Re}(s)\ll 1$, *denote*

$$
\widehat{\mathcal{I}}_{\mathrm{b}}(s):=\int_{\mathbb{A}^\times}\int_{\mathbb{A}^2}\widehat{\Phi}\left(\begin{pmatrix}1 & n_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ z & 0\end{pmatrix}\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\right)\psi(n_1+n_2)\omega(z)|z|^{2-2s}\,\mathrm{d}^\times z
$$

dual to $\mathcal{I}(s)$, *then we have*

(1) When $\mathrm{Re}(s)\gg 1$ *and* $\chi=\omega$,

$$
\sum_{e_1,e_2\in\mathcal{B}(\omega,1)}\mathrm{Res}_{z=s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)
$$
$$
=\mathcal{I}_{\mathrm{b}}(s).
$$

(2) When $\mathrm{Re}(s)\ll 1$ *and* $\chi=\omega$,

$$
\sum_{e_1,e_2\in\mathcal{B}(\omega,1)}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s+\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)
$$
$$
=-\widehat{\mathcal{I}}_{\mathrm{b}}(s).
$$

(3) When $\mathrm{Re}(s)\gg 1$ *and* $\chi=1$, *we have*

$$
\sum_{e_1,e_2\in\mathcal{B}(1,\omega^{-1})}\mathrm{Res}_{z=-s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(-s-\frac{1}{2}\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s+\frac{1}{2}\right)(\mathrm{I}_2)
$$
$$
=-\mathcal{I}_{\mathrm{b}}(s).
$$

(4) When $\mathrm{Re}(s)\ll 1$ *and* $\chi=1$,

$$
\sum_{e_1,e_2\in\mathcal{B}(\omega,1)}\mathrm{Res}_{z=-s+\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(-s+\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s-\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)
$$
$$
=\widehat{\mathcal{I}}_{\mathrm{b}}(s).
$$

Proof. We start with (1). When $\mathrm{Re}(s) \gg 1$, from [Tat67] we know

$$\begin{aligned}\mathrm{Res}_{z=s+\frac{1}{2}} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) &= \int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2}(e_1^\vee \Phi_{e_2}) \left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) \omega(t)|t|^{2s+1} \,\mathrm{d}^\times t \\ &= f_{e_1^\vee \Phi_{e_2}}(\mathrm{I}_2, \mathrm{I}_2; \omega|\cdot|^{2s+\frac{3}{2}}, *).\end{aligned}$$

Consider the decomposition

$$\begin{aligned}&f_\Phi(g_1, g_2; \omega|\cdot|^{2s+\frac{3}{2}}, *) \\ &= \sum_{e_1 \in \mathcal{B}(\chi,1)} \sum_{e_2 \in \mathcal{B}(\chi,1)} \int_{\mathrm{K}^2} f_\Phi(k_1, k_2) \,\mathrm{d}k_1 \,\mathrm{d}k_2 e_1^\vee(k_1) e_2(k_2) e_1\left(s+\frac{1}{2}\right)(g_1) e_2^\vee\left(-s-\frac{1}{2}\right)(g_2).\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\mathrm{K}^2} f_\Phi(k_1, k_2) e_1^\vee(k_1) e_2(k_2) \,\mathrm{d}k_1 \,\mathrm{d}k_2 \\ &= \int_{\mathrm{K}^2} \int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2} \mathrm{L}(k_1) \mathrm{L}(k_2) \Phi\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) \omega(t)|t|^{2s+1} \,\mathrm{d}^\times t e_1^\vee(k_1) e_2(k_2) \,\mathrm{d}k_1 \,\mathrm{d}k_2 \\ &= \int_{\mathbb{A}^\times} \int_{\mathrm{K}^2} \mathbb{F}_{\psi,2} \mathrm{L}(k_1) \mathrm{R}(k_2) \Phi\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) e_1^\vee(k_1) e_2(k_2) \,\mathrm{d}k_1 \,\mathrm{d}k_2 \omega(t)|t|^{2s+1} \,\mathrm{d}^\times t \\ &= \int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2}(e_1^\vee \Phi_{e_2}) \left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) \omega(t)|t|^{2s+1} \,\mathrm{d}^\times t,\end{aligned}$$

we have the Whittaker function of $f_\Phi(g_1, g_2; \omega|\cdot|^{2s+\frac{3}{2}}, *)$ at $(\mathrm{I}_2, \mathrm{I}_2)$ is

$$\sum_{e_1, e_2 \in \mathcal{B}(\chi,1)} f_{e_1^\vee \Phi_{e_2}}(\mathrm{I}_2, \mathrm{I}_2; \omega|\cdot|^{2s+\frac{3}{2}}, *) \mathbb{W}^{\psi^{-1}}\left(s+\frac{1}{2}, e_1\right)(\mathrm{I}_2) \mathbb{W}^\psi\left(-s-\frac{1}{2}, e_2^\vee\right)(\mathrm{I}_2).$$

On the other hand, according to [Wu22, Lemma 4.9], this Whittaker function is given by the integral

$$\int_{\mathbb{A}^\times} \int_{\mathbb{A}^2} \Phi\left(\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}\right) \psi^{-1}(n_1 + n_2) \,\mathrm{d}n_1 \,\mathrm{d}n_2 \omega(z)|z|^{2s+2} \,\mathrm{d}^\times z,$$

which is exactly $\mathcal{I}_{\mathrm{b}}(s)$.

For (3),the residue is given by the following integral when $\mathrm{Re}(s) \gg 1$

$$\begin{aligned}\mathrm{Res}_{z=-s-\frac{1}{2}} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_z(e_2, e_1^\vee)\right) &= -\int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2}(e_1^\vee \Phi_{e_2}) \left(\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\right) |t|^{2s+1} \omega(t) \,\mathrm{d}^\times t \\ &= -f_{e_1^\vee \Phi_{e_2}}(\mathrm{I}_2, \mathrm{I}_2; \omega|\cdot|^{2s+\frac{3}{2}}, \star).\end{aligned}$$

Similarly, we have

$$\sum_{e_1,e_2\in\mathcal{B}(1,\omega^{-1})} f_\Phi(\mathrm{I}_2,\mathrm{I}_2;\omega|\cdot|^{2s+\frac{3}{2}},\star)\mathbb{W}^{\psi^{-1}}\left(-s-\frac{1}{2}\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(s+\frac{1}{2}\right)(\mathrm{I}_2)$$

is the Whittaker function of $f_\Phi(g_1,g_2;\omega|\cdot|^{2s+\frac{3}{2}},\star)$ at $(\mathrm{I}_2,\mathrm{I}_2)$. But according to Lemma 3.4.8 and [GJ79], the intertwining operator does not change the Eisenstein series and the Whittaker functions. So the value of the Whittaker function of $f_\Phi(g_1,g_2;\omega|\cdot|^{2s+\frac{3}{2}},\star)$ at $(\mathrm{I}_2,\mathrm{I}_2)$ is also the value of the Whittaker function of $f_\Phi(g_1,g_2;\omega|\cdot|^{2s+\frac{3}{2}},*)$, which is $\mathcal{I}_{\mathrm{b}}(s)$ according to part (1).

For (2), when $\mathrm{Re}(s)\ll 1$, using the functional equation of GL_1 zeta integrals, we have

$$\begin{aligned}&\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee\right)\\&=-\int_{\mathbb{A}^\times}\mathbb{F}_{\psi,1}\mathbb{F}_{\psi,4}\mathbb{F}_{\psi,2}(e_1^\vee\Phi_{e_2})\left(\begin{pmatrix}t&0\\0&0\end{pmatrix}\right)\omega(t)|t|^{1-2s}\,\mathrm{d}^\times t.\end{aligned}$$

Note that for any $g_1,g_2\in\mathrm{GL}_2(\mathbb{A})$, we have

$$\mathbb{F}_{\psi,2}\mathrm{L}(g_1)\mathrm{R}(g_2)\widehat{\Phi}=\mathbb{F}_{\psi,2}\widehat{\mathrm{L}(g_2)\mathrm{R}(g_1)}\Phi|\det g_1^{-1}g_2|^2.$$

Using the Fourier inversion formula, we have

$$\mathbb{F}_{\psi,2}\widehat{\mathrm{L}(g_2)\mathrm{R}(g_1)}\Phi\left(\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}\right)=\mathbb{F}_{\psi,2}\mathbb{F}_{\psi,1}\mathbb{F}_{\psi,4}\mathrm{L}(g_2)\mathrm{R}(g_1)\Phi\left(\begin{pmatrix}x_{11}&x_{12}\\-x_{21}&x_{22}\end{pmatrix}\right).$$

Then, according to a similar argument, we have

$$\begin{aligned}&\sum_{e_1,e_2\in\mathcal{B}(\omega,1)}\mathrm{Res}_{z=s-\frac{1}{2}}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_z(e_2,e_1^\vee)\right)\mathbb{W}^{\psi^{-1}}\left(s-\frac{1}{2},e_1\right)(\mathrm{I}_2)\mathbb{W}^{\psi}\left(-s+\frac{1}{2},e_2^\vee\right)(\mathrm{I}_2)\\&=\int_{\mathbb{A}^\times}\int_{\mathbb{A}^2}\widehat{\Phi}\left(\begin{pmatrix}1&n_1\\0&1\end{pmatrix}\begin{pmatrix}0&0\\z&0\end{pmatrix}\begin{pmatrix}1&n_2\\0&1\end{pmatrix}\right)\psi(n_1+n_2)\omega(z)|z|^{2-2s}\,\mathrm{d}^\times z.\end{aligned}$$

Finally (4) is a combination of the above (1), (2) and (3), and we leave the verification to the readers. $\square$

Remark 3.4.10. *From the above Proposition, the residues are noting but the Whittaker function of the kernel function given by the summation over the rank 1 rational matrices. In this sense, the boundary term appears as the residue of the Eisenstein series in the spectrum decomposition of the kernel function via analytic continuation.*

Let us then study the analytic continuation of the boundary term $\mathcal{I}_{\mathrm{b}}(s)$. A first observation is that when $\mathrm{Re}(s) \gg 1$,

$$\int_{[\mathrm{N}]} \mathrm{K}'(x,n)\psi^{-1}(n)\,\mathrm{d}n = \int_{\mathrm{N}(\mathbb{A})} \sum_{\gamma\in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \mathrm{R}(\gamma x, wn)\psi^{-1}(n)\,\mathrm{d}n$$

$$= \int_{\mathrm{N}(\mathbb{A})} \sum_{\gamma\in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \int_{\mathbb{A}^\times} \Phi\left(x^{-1}\gamma^{-1}\begin{pmatrix}0 & z\\ 0 & 0\end{pmatrix} wn\right)\psi^{-1}(n)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z|\det x^{-1}|^{s+1}\,\mathrm{d}n$$

$$= \sum_{\gamma\in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \int_{\mathbb{A}}\int_{\mathbb{A}^\times} \mathrm{L}_{\gamma x}\Phi\left(\begin{pmatrix}z & zn\\ 0 & 0\end{pmatrix}\right)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z|\det x^{-1}|^{s+1}\psi^{-1}(n)\,\mathrm{d}n$$

$$= \sum_{\gamma\in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2}\mathrm{L}_{\gamma x}\Phi\left(\begin{pmatrix}z & z^{-1}\\ 0 & 0\end{pmatrix}\right)\omega(z)|z|^{2s+1}\,\mathrm{d}^\times z|\det x^{-1}|^{s+1}.$$

Note that the inner integral is convergent for all $s\in\mathbb{C}$. Now consider the section

$$\mathrm{GL}_2(\mathbb{A})\times\mathbb{C}\to\mathbb{C}:$$

$$(g,s)\mapsto \mathfrak{f}_s(g) := \int_{\mathbb{A}^\times} \mathbb{F}_{\psi,2}\mathrm{L}_g\Phi\left(\begin{pmatrix}z & z^{-1}\\ 0 & 0\end{pmatrix}\right)\omega(z)|z|^{2s+1}\,\mathrm{d}^\times z|\det g^{-1}|^{s+1}.$$

This is a section to the induced principal series for all $s\in\mathbb{C}$, and let

$$\mathrm{E}(\mathfrak{f}_s)(g) := \sum_{\gamma\in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \mathfrak{f}_s(\gamma g)$$

be the corresponding Eisenstein series. When $\mathrm{Re}(s)\gg 1$, we have

$$\int_{[\mathrm{N}]} \mathrm{K}'(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 = \int_{[\mathrm{N}]} \mathrm{E}(\mathfrak{f}_s)(n)\psi(n)\,\mathrm{d}n = \int_{N(\mathbb{A})} \mathfrak{f}_s(wn)\psi(n)\,\mathrm{d}n.$$

Then, the analytic continuation comes from the general theory of the Eisenstein series.

Note that we have

$$\mathbb{F}_{\psi,2}\mathrm{L}_x\Phi\left(\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\right) = \int_{\mathbb{A}} \mathrm{L}_x\Phi\left(\begin{pmatrix}x_{11} & y\\ x_{21} & x_{22}\end{pmatrix}\right)\psi^{-1}(yx_{22})\,\mathrm{d}y.$$

Then for $x = w\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\in \mathrm{GL}_2(\mathbb{A})$, we have

$$\mathbb{F}_{\psi,2}\mathrm{L}_x\Phi\left(\begin{pmatrix}z & z^{-1}\\ 0 & 0\end{pmatrix}\right) = \int_{\mathbb{A}} \Phi\left(\begin{pmatrix}-nz & -ny\\ z & y\end{pmatrix}\right)\psi^{-1}(yz^{-1})\,\mathrm{d}y.$$

Then integral over n and make a change of variable by $y \mapsto zn_1, n \mapsto -n_2$, we have

$$\begin{aligned}&\int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}'(n_1, n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\\&= \int_{\mathbb{A}^2}\int_{\mathbb{A}^\times} \Phi\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi^{-1}(n_1+n_2)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2.\end{aligned}$$

This also coincides with the calculation in Section 3.3.1.

Note that from here we can see that $\mathcal{I}_{\mathrm{b}}(s)$ admits an Euler product

$$\prod_\nu \int_{k_\nu^2}\int_{k_\nu^\times} \Phi_\nu\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi_\nu^{-1}(n_1+n_2)\omega_\nu(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

Let us finish this section with the unramified calculation.

Proposition 3.4.11. *Let $\Phi_\nu = 1_{\mathrm{Mat}_2(\mathfrak{o}_\nu)}$, and ψ_ν is unramified, then we have*

$$\begin{aligned}&\int_{k_\nu^2}\int_{k_\nu^\times} \Phi_\nu\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi_\nu^{-1}(n_1+n_2)\omega_\nu(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2\\&= 1-\omega_\nu(\varpi_\nu)q^{-(2s+2)}\end{aligned}$$

if we normalize the measure $\mathrm{d}^\times z$ such that $\mathrm{vol}(\mathfrak{o}_\nu^\times, \mathrm{d}^\times z) = 1$. So let S be a finite set of places of k containing all the Archimedean places and such that for all $v \notin S$ we have $\Phi_\nu = 1_{\mathrm{Mat}_2(\mathfrak{o}_\nu)}$, ψ_ν is unramified and $\mathrm{vol}(\mathfrak{o}_\nu^\times, \mathrm{d}^\times z) = 1$, then we have

$$\begin{aligned}&\prod_{\nu\notin S} \int_{k_\nu^2}\int_{k_\nu^\times} \Phi_\nu\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi_\nu^{-1}(n_1+n_2)\omega_\nu(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2\\&= \frac{1}{L^S(\omega, 2s+2)},\end{aligned}$$

where $L^S(\omega, s)$ is the partial Dirichlet L-function.

Proof. A first observation is that

$$\begin{aligned}&\int_{k_\nu^2}\int_{k_\nu^\times} \Phi_\nu\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi^{-1}(n_1+n_2)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2\\&= \sum_{m=0}^\infty \int_{\varpi_\nu^m\mathfrak{o}_\nu^\times} \omega_\nu(\varpi_\nu)^m q_\nu^{-m(2s+2)} \int_{|zn_1|\le 1, |zn_2|\le 1, |zn_1n_2|\le 1} \psi_\nu^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}^\times z.\end{aligned}$$

We first fix z such that $z \in \varpi_\nu^m \mathfrak{o}_\nu^\times$, and then $n_1 \in \varpi_\nu^{m_1}\mathfrak{o}_\nu^\times$ with $m + m_1 \geq 0$. Then the condition for n_2 is

$$|n_2| \leq \min\{q^m, q^{m+m_1}\}.$$

Therefore, the inner integral becomes

$$\begin{aligned}
&\int_{|zn_1|\leq 1, |zn_2|\leq 1, |zn_1n_2|\leq 1} \psi_\nu^{-1}(n_1+n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2 \\
&= \sum_{m_1=-m}^{0} \int_{n_1\in\varpi_\nu^{m_1}\mathfrak{o}_\nu^\times} \psi_\nu^{-1}(n_1)\,\mathrm{d}n_1 \int_{|n_2|\leq q^{m+m_1}} \psi_\nu^{-1}(n_2)\,\mathrm{d}n_2 \\
&+ \sum_{m_1=1}^{\infty} \int_{n_1\in\varpi_\nu^{m_1}\mathfrak{o}_\nu^\times} \psi_\nu^{-1}(n_1)\,\mathrm{d}n_1 \int_{|n_2|\leq q^{m}} \psi_\nu^{-1}(n_2)\,\mathrm{d}n_2 \\
&= \int_{n_1\in\varpi_\nu^{-m}\mathfrak{o}_\nu^\times} \psi_\nu^{-1}(n_1)\,\mathrm{d}n_1 + q^{-1}\int_{|n_2|\leq q^{m}} \psi_\nu^{-1}(n_2)\,\mathrm{d}n_2 \\
&= \begin{cases} 1 & \text{if} \quad m=0 \\ -1 & \text{if} \quad m=1 \\ 0 & \text{if} \quad m\geq 2 \end{cases}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{k_\nu^2}\int_{k_\nu^\times} \Phi_\nu\left(\begin{pmatrix} zn_2 & zn_1n_2 \\ z & zn_1 \end{pmatrix}\right)\psi^{-1}(n_1+n_2)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z\,\mathrm{d}n_1\,\mathrm{d}n_2 \\
&= 1-\omega_\nu(\varpi_\nu)q^{-(2s+2)}.
\end{aligned}$$

□

Corollary 3.4.12. *The cuspidal part $\mathcal{I}_{\mathrm{cusp}}(s)$ admits an analytic continuation to all $s \in \mathbb{C}$.*

3.4.4 Functional Equations in the Trace Formula

In this section, we will explain that the Theorem 3.3.6 is responsible for the functional equation of standard L-functions of GL_2. Let Φ be a Schwartz function on $\mathrm{Mat}_2(\mathbb{A})$ as before and $\widehat{\Phi}$ its Fourier transform. Denote for $x, y \in \mathrm{Mat}_2^\vee(\mathbb{A})$

$$\widehat{\mathrm{K}}(x,y) := \int_{[\mathrm{Z}]} \sum_{\gamma\in\mathrm{GL}_2(k)} \widehat{\Phi}(zy^{-1}\gamma x)\omega^{-1}(z)|z|^{-2s+2}\,\mathrm{d}^\times z \cdot |\det y^{-1}x|^{-s+1},$$

and

$$\widehat{\mathrm{K}}'(x,y) := \int_{[\mathrm{Z}]} \sum_{\gamma\in \mathrm{Mat}_2(k),\mathrm{rank}\gamma=1} \widehat{\Phi}(zy^{-1}\gamma x)\omega^{-1}(z)|z|^{-2s+2}\,\mathrm{d}^\times z\cdot |\det y^{-1}x|^{-s+1}.$$

Let

$$\widehat{\mathcal{I}}(s) := \int_{[\mathrm{Z}]} \mathrm{KTF}(\mathrm{R}_z(\widehat{\Phi}(x)(\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z,$$

and

$$\widehat{\mathcal{I}}_{\mathrm{cusp}}(s) := \sum_{\pi \text{ cuspidal},\omega_\pi=\omega}\ \sum_{\varphi_1.\varphi_2\in\mathcal{B}(\pi)} \mathcal{Z}\left(-s+\frac{1}{2},\widehat{\Phi},\beta^\vee(\varphi_2,\varphi_1^\vee)\right)\mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2).$$

It is also easy to see that $\widehat{\mathcal{I}}_{\mathrm{b}}(s)$ defined in Proposition 3.4.9 is

$$\widehat{\mathcal{I}}_{\mathrm{b}}(s) = \int_{[\mathrm{N}]\times[\mathrm{N}]} \widehat{\mathrm{K}}'(n_1,n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

By identifying $\mathrm{Mat}_2^\vee$ with Mat_2 and replace Φ by $\widehat{\Phi}$ in Proposition 3.4.4, we see $\widehat{\mathcal{I}}(s),\widehat{\mathcal{I}}_{\mathrm{Eis}}(s)$ and $\widehat{\mathcal{I}}_{\mathrm{cusp}}(s)$ also admit meromorphic continuations to all $s\in\mathbb{C}$. Then according to Theorem 3.3.6 again, we have

Proposition 3.4.13. *$\mathcal{I}(s)=\widehat{\mathcal{I}}(s)$ for all $s\in\mathbb{C}$.*

Proof. Because both of them are

$$\int_{z\in[\mathrm{Z}],|z|\ge 1} \mathrm{KTF}(\mathrm{R}_z(\Phi(x)(\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z + \int_{|z|<1} \mathrm{KTF}(\mathrm{R}_z(\widehat{\Phi}(x)(\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z$$

as in the proof of Proposition 3.4.4. □

According to [Wu22, Theorem 2.10] by considering the action of $\widehat{\Phi}(g^{-1})|\det g|^{s-1}$ on $(\mathrm{R}_\omega,\mathcal{L}^2(\mathrm{GL}_2,\omega))$, we have the spectrum decomposition of $\widehat{\mathrm{K}}(x,y)$:

$$\begin{aligned}\widehat{\mathrm{K}}(x,y) = &\sum_{\pi \text{ cuspidal},\omega_\pi=\omega}\ \sum_{\varphi_1.\varphi_2\in\mathcal{B}(\pi)} \mathcal{Z}\left(-s+\frac{1}{2},\widehat{\Phi},\beta^\vee(\varphi_2,\varphi_1^\vee)\right)\varphi_1(x)\varphi_2^\vee(y)\\ &+\sum_{\chi\in\mathbb{R}_+\widehat{k^\times\backslash\mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})} \mathcal{Z}\left(-s+\frac{1}{2},\widehat{\Phi},\beta_{i\tau}^\vee(e_2,e_1^\vee)\right)\mathrm{E}(i\tau,e_1)(x)\mathrm{E}(-i\tau,e_2^\vee)(y)\frac{\mathrm{d}\tau}{4\pi}\\ &+\frac{1}{\mathrm{vol}[\mathrm{PGL}_2]}\sum_{\eta\in\widehat{k^\times\backslash\mathbb{A}^\times},\eta^2=\omega} \left(\Phi(g)\eta(\det g)|\det g|^{-s+1}\,\mathrm{d}g\right)\overline{\eta(\det x)}\eta(\det y),\end{aligned}$$

and integrate it against the same character over $[\mathrm{N}] \times [\mathrm{N}]$, we obtain

$$\int_{[\mathrm{N}]\times[\mathrm{N}]} \widehat{\mathrm{K}}(n_1, n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2$$
$$= \sum_{\pi \text{ cuspidal}, \omega_\pi=\omega} \sum_{\varphi_1.\varphi_2\in\mathcal{B}(\pi)} \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta^\vee(\varphi_2, \varphi_1^\vee)\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2)$$
$$+ \sum_{\chi\in\mathbb{R}_+\widehat{k^\times\backslash\mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})} \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta_{i\tau}(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(i\tau, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}.$$

Then when $\mathrm{Re}(s) < -1$ we have

$$\widehat{\mathcal{I}}(s) = \int_{Z(k)\backslash Z(\mathbb{A})} \mathrm{KTF}(\mathrm{R}_z(\widehat{\Phi}(x)(\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z$$
$$= \int_{[\mathrm{N}]\times[\mathrm{N}]} \left(\widehat{\mathrm{K}}(n_1, n_2) + \widehat{\mathrm{K}}'(n_1, n_2)\right)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_2\,\mathrm{d}n_2$$
$$= \sum_{\pi \text{ cuspidal}, \omega_\pi=\omega} \sum_{\varphi_1.\varphi_2\in\mathcal{B}(\pi)} \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta^\vee(\varphi_2, \varphi_1^\vee)\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2)\mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2)$$
$$+ \sum_{\chi\in\mathbb{R}_+\widehat{k^\times\backslash\mathbb{A}^\times}} \int_{-\infty}^{\infty} \sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})} \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta_{i\tau}^\vee(e_2, e_1^\vee)\right) \mathbb{W}^{\psi^{-1}}(i\tau, e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau, e_2^\vee)(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}$$
$$+ \int_{[\mathrm{N}]\times[\mathrm{N}]} \widehat{\mathrm{K}}'(n_1, n_2)\psi^{-1}(n_1^{-1}n_2)\,\mathrm{d}n_1\,\mathrm{d}n_2.$$

Because

$$\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_{i\tau}(e_2, e_1^\vee)\right)$$
$$= \int_{(\mathbb{A}^\times)^2} \mathbb{F}_{\psi,2}({}_{e_1^\vee}\Phi_{e_2})\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \chi(t_1)|t_1|^{s+\frac{1}{2}+i\tau}\chi^{-1}(t_2)|t_2|^{s+\frac{1}{2}-i\tau}\,\mathrm{d}^\times t_1\,\mathrm{d}^\times t_2,$$

and

$$\mathbb{F}_{\psi,2}(({}_{e_1^\vee}\widehat{\Phi}_{e_2})) = \mathbb{F}_{\psi,2}((\widehat{{}_{e_2}\Phi_{e_1^\vee}})) = \mathbb{F}_{\psi,1}\mathbb{F}_{\psi,4}\mathbb{F}_{\psi,2}({}_{e_2}\Phi_{e_1^\vee})\begin{pmatrix} \cdot & \cdot \\ -\cdot & \cdot \end{pmatrix},$$

we can apply the functional equation of GL_1 zeta integrals to obtain

$$\mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta_{i\tau}^\vee(e_2, e_1^\vee)\right)$$
$$= \int_{(\mathbb{A}^\times)^2} \mathbb{F}_{\psi,1}\mathbb{F}_{\psi,4}\mathbb{F}_{\psi,2}({}_{e_2}\Phi_{e_1^\vee})\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \chi(t_1)|t_1|^{-s+\frac{1}{2}+i\tau}\chi^{-1}(t_2)|t_2|^{-s+\frac{1}{2}-i\tau}\,\mathrm{d}^\times t_1\,\mathrm{d}^\times t_2$$
$$= \int_{(\mathbb{A}^\times)^2} \mathbb{F}_{\psi,2}({}_{e_2}\Phi_{e_1^\vee})\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \chi^{-1}(t_1)|t_1|^{s+\frac{1}{2}-i\tau}\chi(t_2)|t_2|^{s+\frac{1}{2}+i\tau}\,\mathrm{d}^\times t_1\,\mathrm{d}^\times t_2$$
$$= \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta_{-i\tau}(e_1^\vee, e_2)\right).$$

Therefore

$$\sum_{\chi\in\widehat{\mathbb{R}_+k^\times\backslash\mathbb{A}^\times}}\int_{-\infty}^{\infty}\sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})}\mathcal{Z}\left(-s+\frac{1}{2},\widehat{\Phi},\beta_{i\tau}^{\vee}(e_2,e_1^{\vee})\right)\mathbb{W}^{\psi^{-1}}(i\tau,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau,e_2^{\vee})(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}$$
$$=\sum_{\chi\in\widehat{\mathbb{R}_+k^\times\backslash\mathbb{A}^\times}}\int_{-\infty}^{\infty}\sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_{-i\tau}(e_1^{\vee},e_2)\right)\mathbb{W}^{\psi^{-1}}(i\tau,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau,e_2^{\vee})(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}.$$

By making $\tau\mapsto-\tau$ and change the role of e with $e^{\vee}$, we have the above is

$$\sum_{\chi\in\widehat{\mathbb{R}_+k^\times\backslash\mathbb{A}^\times}}\int_{-\infty}^{\infty}\sum_{e_1,e_2\in\mathcal{B}(\chi,\omega\chi^{-1})}\mathcal{Z}\left(s+\frac{1}{2},\Phi,\beta_{i\tau}(e_2,e_1^{\vee})\right)\mathbb{W}^{\psi^{-1}}(i\tau,e_1)(\mathrm{I}_2)\mathbb{W}^{\psi}(-i\tau,e_2^{\vee})(\mathrm{I}_2)\frac{\mathrm{d}\tau}{4\pi}$$
$$=\mathcal{I}(s)-\left(-\frac{1}{2}\mathcal{I}_{\mathrm{b}}(s)\right)+\left(\frac{1}{2}\mathcal{I}_{\mathrm{b}}(s)\right)-\frac{1}{2}\widehat{\mathcal{I}}_{\mathrm{b}}(s)+\left(-\frac{1}{2}\widehat{\mathcal{I}}_{\mathrm{b}}(s)\right),$$

i.e.,

$$\widehat{\mathcal{I}}_{\mathrm{Eis}}(s)=\mathcal{I}_{\mathrm{Eis}}(s)+\mathcal{I}_{\mathrm{b}}(s)-\widehat{\mathcal{I}}_{\mathrm{b}}(s).$$

Combining with Proposition 3.4.13 and Theorem 3.4.1, we obtain the functional equation of the cuspidal parts:

Corollary 3.4.14.

$$\mathcal{I}_{\mathrm{cusp}}(s)=\widehat{\mathcal{I}}_{\mathrm{cusp}}(s).$$

3.4.5 Isolation of the Spectrum

Let S be a finite set of places of k containing all the Archimedean places. Denote by $\mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))_{\mathrm{S}}$ the subspace of Schwartz functions on $\mathrm{Mat}_2(\mathbb{A})$ consisting of vectors of the form

$$\Phi_{\mathrm{S}}\otimes_{\nu\notin\mathrm{S}}1_{\mathrm{Mat}_2(\mathfrak{o}_\nu)}.$$

For simplicity, denote $\Phi^{\circ,\mathrm{S}}:=\otimes_{\nu\notin\mathrm{S}}1_{\mathrm{Mat}_2(\mathfrak{o}_\nu)}$ and write Φ_{S} as elements in $\mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))_{\mathrm{S}}$ by tensoring this basic vector. Let $\mathcal{H}^{\mathrm{S}}$ be the unramified Hecke algebra outside S

$$\mathcal{H}^{\mathrm{S}}:=\otimes'_{\nu\notin\mathrm{S}}\mathcal{H}(\mathrm{G}(k_\nu),\mathrm{K}_\nu).$$

Then we get a family of functionals first when $\mathrm{Re}(s)>1$

$$\begin{aligned}\mathrm{KTF}^{\mathrm{S}}(s):\mathcal{H}^{\mathrm{S}}\otimes\mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))_{\mathrm{S}}&\to\mathbb{C}\\ h\otimes\Phi_{\mathrm{S}}&\mapsto\int_{[\mathrm{Z}]}\mathrm{KTF}(\mathrm{R}_z(\Phi_{\mathrm{S}}\otimes(h\star\Phi^{\circ,\mathrm{S}})(x)(\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s}\,\mathrm{d}^\times z,\end{aligned}$$

By analytic continuation from Section 3.4.3, we obtain a family $\mathrm{KTF}^{\mathrm{S}}(s)$ for $s \in \mathbb{C}$.

By Satake isomorphism, we have for each non-Archimedean place ν,

$$\mathcal{H}(\mathrm{G}(k_\nu), \mathrm{K}_\nu) \cong \mathbb{C}[\mathrm{G}^\vee /\!/ \mathrm{G}^\vee],$$

where the action is by conjugation. Hence

$$\mathcal{H}^{\mathrm{S}} \cong \mathbb{C}\left[\prod_{\nu \notin \mathrm{S}} (\mathrm{G}^\vee /\!/ \mathrm{G}^\vee)\right],$$

where regular functions are restricted tensor products of regular functions on finite factors with respect to the constant function 1. This isomorphism is denoted by $h \mapsto \widehat{h}$. Since every automorphic representation that is unramified outside of S determines a point on $\prod_{\nu \notin \mathrm{S}} \mathrm{G}^\vee /\!/ \mathrm{G}^\vee$ and is determined by it due to strong multiplicity one. Let

$$\mathrm{U}^{\mathrm{S}} \subset \prod_{\nu \notin \mathrm{S}} \mathrm{G}^\vee /\!/ \mathrm{G}^\vee$$

be the subset corresponding to the unitary ones with given central character ω, which is compact with the product Hausdorff topology. Let

$$\mathrm{R} = \{[\pi(\omega|\cdot|^{s+\frac{1}{2}}, |\cdot|^{-s-\frac{1}{2}})] : s \in \mathbb{C}\}.$$

We write

$$\mathbb{C}[\mathrm{U}^{\mathrm{S}} \cup \mathrm{R}] = \mathbb{C}\left[\prod_{\nu \notin \mathrm{S}} (\mathrm{G}^\vee /\!/ \mathrm{G}^\vee)\right],$$

by restriction. Then we have

Lemma 3.4.15. $\mathbb{C}[\mathrm{U}^{\mathrm{S}} \cup \mathrm{R}]$ *is dense on the space* $\mathbb{C}(\mathrm{U}^{\mathrm{S}} \cup \mathrm{R})$ *of continuous functions on* $\mathrm{U}^{\mathrm{S}} \cup \mathrm{R}$*, where the latter is equipped with the inductive limit topology of uniform convergence on compact subsets.*

Proof. According to the choice of the topology here, it suffices to show for any compact subset C, the restriction of $\mathbb{C}[\mathrm{U}^{\mathrm{S}} \cup \mathrm{R}]$ to C is dense in $\mathbb{C}(\mathrm{C})$, and it suffices to show polynomial functions in $\mathbb{C}[\mathrm{U}^{\mathrm{S}} \cup \mathrm{R}]$ are closed under complex conjugation, which is [Sak19, Lemma 5.1.1]. □

Fix an element $\Phi_{\mathrm{S}} \in \mathcal{F}(\mathrm{Mat}_2(\mathbb{A}))_{\mathrm{S}}$, then we can view $\mathrm{KTF}^{\mathrm{S}}(s)$ as a functional on $\mathcal{H}^{\mathrm{S}}$, which can be extended to a continuous functional on $\mathbb{C}(\mathrm{U}^{\mathrm{S}} \cup \mathrm{R})$, hence a finite measure with compact support:

Lemma 3.4.16. $\mathrm{KTF}^{\mathrm{S}}(s)$ *can be extended to a finite measure with compact support on* $\mathbb{C}(\mathrm{U}^{\mathrm{S}} \cup \mathrm{R})$.

Proof. To avoid confusion, for a Schwartz function Φ on $\mathrm{Mat}_2(\mathbb{A})$, denote $\mathrm{K}_\Phi(x,y)$ to be the kernel function associated to Φ defined by

$$\mathrm{K}_\Phi(x,y) := \int_{[\mathrm{Z}]} \sum_{\gamma \in \mathrm{GL}_2(k)} \Phi(x^{-1}\gamma y z)\omega(z)|z|^{2s+2}\,\mathrm{d}^\times z \cdot |\det x^{-1}y|^{s+1}$$

as in the previous sections. And similarly for $\mathrm{K}'_\Phi(x,y)$, $\widehat{\mathrm{K}}_{\widehat{\Phi}}(x,y)$ and $\widehat{\mathrm{K}}'_{\widehat{\Phi}}(x,y)$.

When $\mathrm{Re}(s) > 1$, according to [Wu17, Lemma 3.25], we can choose some compact subset Ω of $\mathrm{G}(\mathbb{A})$ containing a fundamental domian of $[\mathrm{N}]$, then we have for $y \in \Omega$

$$\begin{aligned} &|\mathrm{K}_{\Phi_{\mathrm{S}} \otimes h \star \Phi^{\circ,\mathrm{S}}}(x,y)| \\ &\ll \left(\|(1+\Delta_\infty)^{N+A}\mathrm{K}_\Phi(x,\cdot)\|_{\mathcal{L}^2(\mathrm{GL}_2,\omega)} + \sum_{\chi^2=\omega} \|\mathrm{F}_{\chi\circ\det}(\mathrm{K}(x,\cdot))\| \right) \cdot \|\widehat{h}\|_{\mathcal{L}^\infty(\mathrm{U}^{\mathrm{S}})}, \end{aligned} \tag{3.6}$$

where $\Phi = \Phi_{\mathrm{S}} \otimes \Phi^{\circ,\mathrm{S}}$, $N \in \mathbb{N}$ depends only on the field k, A is some real number large enough, Δ_∞ is the Laplacian operator defined as in [Wu17, Section 3.5], and $\mathrm{F}_{\chi\circ\det}$ is the spectrum projectors defined as in [Wu17, Section 2.2]. Note that the right-hand-side of (3.6) is a continuous function for the x-variable, then after we integrate it over $[\mathrm{N}]$, we see

$$\int_{[\mathrm{N}]\times[\mathrm{N}]} \mathrm{K}(n_1,n_2)\psi^{-1}(n_1^{-1}n_2) \ll \|\widehat{h}\|_{\mathcal{L}^\infty(\mathrm{U}^{\mathrm{S}})},$$

where the implict constant depends on Φ_{S}, s and the underground field k. Also note that according to [Wu22, Theorem 2.10], we have

$$(1+\Delta_\infty)^{N+A}\mathrm{K}_\Phi(x,y) \ll \mathrm{Ht}(y)^{1-s},$$

where the hight function Ht is as defined in [Wu22, Section 1.2]. Then, when $\mathrm{Re}(s)$ is fixed, the implicit constant can be chosen to be independent of s.

According to Section 3.4.2, as the kernel function $\mathrm{K}'_{\Phi_{\mathrm{S}} \otimes (h \star \Phi^{\circ,\mathrm{S}})}(x,y)$ is an Eisenstein series and the action of h is given by the value of the delta distribution of $\widehat{h}$ on the principal series $\pi(\omega|\cdot|^{s+\frac{1}{2}}, |\cdot|^{-s-\frac{1}{2}})$, we get the desired extension when $\mathrm{Re}(s) > 1$.

When $\mathrm{Re}(s) < -1$, combine Proposition 3.4.13 and the fundamental lemma Proposition 3.2.35, we have

$$\mathrm{KTF}^{\mathrm{S}}(s)(h) \ll \|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{U}^{\mathrm{S}})} + |\widehat{h}(\pi(\omega|\cdot|^{s-\frac{1}{2}}, |\cdot|^{-s+\frac{1}{2}}))|, \tag{3.7}$$

which means $\mathrm{KTF}^{\mathrm{S}}(s)$ is also a finite measure of compact support.

When $-1 \leq \mathrm{Re}(s) \leq 1$, we will use the Phragmen-Lindelof principle to bound $\mathrm{KTF}^{\mathrm{S}}(s)$. First let us fix $\epsilon > 0$ and consider the case when $\mathrm{Re}(s) = 1+\epsilon$. As $\mathrm{Re}(s) = 1+\epsilon$ is fixed, there is some compact set C' of

$$\prod_{\nu \notin \mathrm{S}} (\mathrm{G}^{\vee} /\!\!/ \mathrm{G}^{\vee})$$

such that C' contains $\{[\pi(\omega|\cdot|^{s+\frac{1}{2}}, |\cdot|^{-s-\frac{1}{2}})] : s \in \mathbb{C}, \mathrm{Re}(s) = 1+\epsilon\}$, hence

$$\|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{U}^{\mathrm{S}})} + |\widehat{h}(\pi(\omega|\cdot|^{s-\frac{1}{2}}, |\cdot|^{-s+\frac{1}{2}}))| \leq \|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{C})}, \ \mathrm{Re}(s) = 1+\epsilon,$$

where $\mathrm{C} := \mathrm{U}^{\mathrm{S}} \cup \mathrm{C}'$ is compact. Moreover, when $\mathrm{Re}(s) = 1+\epsilon$, the growth of

$$(1+\Delta_{\infty})^{N+A}\mathrm{K}_{\Phi}(x,\cdot)$$

on any Seigel domain is bounded by $\mathrm{Ht}(y)^{-\epsilon}$, so are

$$\|(1+\Delta_{\infty})^{N+A}\mathrm{K}_{\Phi}(x,\cdot)\|_{\mathcal{L}^{2}(\mathrm{GL}_2,\omega)} \text{ and } \|\mathrm{F}_{\chi\circ\det}(\mathrm{K}(x,\cdot))\|.$$

Combine them together, we have

$$|\mathrm{KTF}^{\mathrm{S}}(s)(h)| \ll \|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{C})},$$

when $\mathrm{Re}(s) = 1+\epsilon$, with the implicit constant indepdent of s, i.e., there is some constant B' such that $|\mathrm{KTF}^{\mathrm{S}}(s)(h)| \leq \mathrm{B}'\|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{C})}$ for any s with $\mathrm{Re}(s) = 1+\epsilon$. A similar argument applied to the case that $\mathrm{Re}(s) = -1-\epsilon$ due to Theorem 3.3.6, with replacing Φ by $\widehat{\Phi}$. Then we obtain that there is some constant $\mathrm{B} > 0$ such that

$$|\mathrm{KTF}^{\mathrm{S}}(s)(h)| \leq \mathrm{B}\|\widehat{h}\|_{\mathcal{L}^{\infty}(\mathrm{C})}, \ \mathrm{Re}(s) = 1+\epsilon \text{ or } -1-\epsilon.$$

Next let us show that $\mathrm{KTF}^{\mathrm{S}}(s)(h)$ is bounded if h fixed when $-1-\epsilon \leq \mathrm{Re}(s) \leq 1+\epsilon$. According to the Lemma 3.3.5, when $-1-\epsilon \leq \mathrm{Re}(s) \leq 1+\epsilon$, we have

$$\int_{z\in[\mathrm{Z}],|z|\geq 1} \left|\mathrm{KTF}(\mathrm{R}_z(\Phi_{\mathrm{S}} \otimes h \star \Phi^{\circ,\mathrm{S}})(\,\mathrm{d}x)^{\frac{1}{2}})\omega(z)|z|^{2s}\right| \,\mathrm{d}^\times z$$
$$\leq \int_{z\in[\mathrm{Z}],|z|\geq 1} \int_{\mathrm{N}(\mathbb{A})\times\mathrm{N}(\mathbb{A})} \sum_{t_1\in k^\times, t_2\in k} \left|\mathbb{F}_{\psi,2}(\Phi_{\mathrm{S}} \otimes (h\star\Phi^{\circ,\mathrm{S}}))\left(\begin{pmatrix} t_1 n_1 z & \frac{t_2}{z} \\ t_1 z & t_1 n_2 z\end{pmatrix}\right)\right| |z|^{2s+2} \,\mathrm{d}n_1 \,\mathrm{d}n_2 \,\mathrm{d}^\times z$$
$$+ \int_{z\in\mathrm{Z}(\mathbb{A}),|z|\geq 1} \int_{\mathrm{N}(\mathbb{A})} \left|\Phi\left(\begin{pmatrix} z & zn \\ 0 & z\end{pmatrix}\right)\right| |z|^{2s+2} \,\mathrm{d}^\times z.$$

According to Lemma 3.4.3, there is some function $\widetilde{\Phi}'$ of Schwrtz type such that

$$\sum_{,t_2\in k} \left|\mathbb{F}_{\psi,2}(\Phi_{\mathrm{S}} \otimes (h\star\Phi^{\circ,\mathrm{S}}))\left(\begin{pmatrix} t_1 n_1 z & \frac{t_2}{z} \\ t_1 z & t_1 n_2 z\end{pmatrix}\right)\right| \ll (1+|z|)\widetilde{\Phi}'(t_1 z),$$

hence there is another Schwartz function $\widetilde{\Phi}$ such that

$$\int_{z\in[\mathrm{Z}],|z|\geq 1} \left|\mathrm{KTF}(\mathrm{R}_z(\Phi_{\mathrm{S}} \otimes h \star \Phi^{\circ,\mathrm{S}})(\,\mathrm{d}x)^{\frac{1}{2}})\omega(z)|z|^{2s}\right| \,\mathrm{d}^\times z$$
$$\ll \int_{z\in\mathbb{Z}(\mathbb{A}),|z|\geq 1} (1+|z|)\widetilde{\Phi}(z)(|z|^{2s}+|z|^{2s+1}) \,\mathrm{d}^\times z \ll 1,\ -1-\epsilon \leq \mathrm{Re}(s) \leq 1+\epsilon.$$

Apply the same argument to $\widetilde{\Phi}$, and make use of Proposition 3.4.13, we have

$$\mathrm{KTF}^{\mathrm{S}}(s)(h) = \int_{z\in[\mathrm{Z}],|z|\geq 1} \mathrm{KTF}(\mathrm{R}_z((\Phi_{\mathrm{S}} \otimes h\star\Phi^{\circ,\mathrm{S}})(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s} \,\mathrm{d}^\times z$$
$$+ \int_{|z|<1} \mathrm{KTF}(\mathrm{R}_z(\Phi_{\mathrm{S}} \widehat{\otimes h \star} \Phi^{\circ,\mathrm{S}}(x)(\,\mathrm{d}^+x)^{\frac{1}{2}}))\omega(z)|z|^{2s} \,\mathrm{d}^\times z \ll 1,\ -1-\epsilon \leq \mathrm{Re}(s) \leq 1+\epsilon.$$

Finally, fix h and apply the Phragmen-Lindelof principle ([Rud87, Theorem 12.8]) to the holomorphic function $s \mapsto \mathrm{KTF}^{\mathrm{S}}(s)(h)$, we have

$$|\mathrm{KTF}^{\mathrm{S}}(s)(h)| \leq \mathrm{B}\|\widehat{h}\|_{\mathcal{L}^\infty(\mathrm{C})},\ -1-\epsilon \leq \mathrm{Re}(s) \leq 1+\epsilon,$$

which means $\mathrm{KTF}^{\mathrm{S}}(s)$ is also a finite measure when $-1 \leq \mathrm{Re}(s) \leq 1$.

□

We denote $\widehat{\mathrm{G}}^{\mathrm{aut}}$, the set of automorphic representations appearing in the Plancherel formula for $\mathcal{L}^2(\mathrm{GL}_2,\omega)$. We consider $\widehat{\mathrm{G}}^{\mathrm{aut}}$ as a subset of

$$\lim_{\rightarrow} \mathrm{U}^{\mathrm{S}}$$

in the obvious sense, and we freely talk about $\widehat{\mathrm{G}}^{\mathrm{aut}}$ as a subset of U^{S}, meaning its intersection with U^{S}. We denote $\widehat{\mathrm{G}}^{\mathrm{cusp}}$ the subset of $\widehat{\mathrm{G}}^{\mathrm{aut}}$ consisting of cuspidal representations and $\widehat{\mathrm{G}}^{\mathrm{Eis}}$ the subset of principle series. Then $\widehat{\mathrm{G}}^{\mathrm{cusp}}$ is a countable number points and $\widehat{\mathrm{G}}^{\mathrm{Eis}}$ is a countable union of lines. Both of them are measurable sets with respect to the standard Borel structure on U^{S}.

Then, combined with the spectral expansion, we have

Theorem 3.4.17. *For fixed Φ_{S} and all $s \in \mathbb{C}$, The functional $\mathrm{KTF}^{\mathrm{S}}(s)$ is a finite complex measure $\nu_{\Phi_{\mathrm{S}}}$ on $\widehat{\mathrm{G}}^{\mathrm{cusp}} \cup \widehat{\mathrm{G}}^{\mathrm{Eis}} \cup \mathrm{R}$. The measure $\nu_{\Phi_{\mathrm{S}}}$ is equal to*

$$\sum_{\varphi_1 . \varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^{\vee})\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2) \mathbb{W}_{\varphi_2^{\vee}}^{\psi}(\mathrm{I}_2)\, \mathrm{d}\pi$$

on $\widehat{\mathrm{G}}^{\mathrm{cusp}}$ when $\mathrm{Re}(s) > 1$. It admits an analytic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$\begin{aligned}
&\sum_{\varphi_1 . \varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^{\vee})\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2) \mathbb{W}_{\varphi_2^{\vee}}^{\psi}(\mathrm{I}_2) \\
&= \sum_{\varphi_1 . \varphi_2 \in \mathcal{B}(\pi)} \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi}, \beta^{\vee}(\varphi_2, \varphi_1^{\vee})\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2) \mathbb{W}_{\varphi_2^{\vee}}^{\psi}(\mathrm{I}_2).
\end{aligned}$$

In particular, we get the analytic continuation of $L(s, \pi, \mathrm{st})$. Moreover,

$$L\left(s+\frac{1}{2}, \pi, \mathrm{st}\right) = \epsilon\left(s+\frac{1}{2}, \pi\right) L\left(-s+\frac{1}{2}, \pi, \mathrm{st}^{\vee}\right).$$

where $L\left(s+\frac{1}{2}, \pi, \mathrm{st}\right)$ is the complete L-function.

Proof. We only need to prove the analytic continuation and functional equation of L-functions. Write $\pi = \otimes_{\nu}' \pi_{\nu}$ according to [Fla79]. For $\varphi \in \mathcal{B}(\pi)$, write

$$\varphi = \otimes_{\nu \in |k|}' \varphi_{\nu}.$$

Then we have an Euler product

$$\mathcal{Z}\left(s+\frac{1}{2}, \Phi, \beta(\varphi_2, \varphi_1^{\vee})\right) = \prod_{\nu \in |k|} \mathcal{Z}_{\nu}\left(s+\frac{1}{2}, \Phi_{\nu}, \beta(\varphi_{2,\nu}, \varphi_{1,\nu}^{\vee})\right),$$

where

$$\mathcal{Z}_{\nu}\left(s+\frac{1}{2}, \Phi_{\nu}, \beta(\varphi_{2,\nu}, \varphi_{1,\nu}^{\vee})\right) := \int_{\mathrm{G}(k_{\nu})} \Phi_{\nu}(g) \beta(\varphi_{2,\nu}, \varphi_{1,\nu}^{\vee})(g) |\det g|_{\nu}^{s+1}\, \mathrm{d}g.$$

For non-Archimedean ν, let φ_ν° be the unique newvector of π_ν as in [JPSS81, Theorem (5)] up to scalar. For Archimedean ν, let φ_ν° be the unique highest weight vector of π_ν. Denote $\varphi^\circ = \otimes' \varphi_\nu^\circ$. Then choose $\Phi^\circ = \otimes'_\nu \Phi_\nu^\circ$ such that Φ_ν° is as in [Hum21, Theorem 1.2] and as in [Lin18, Theorem 1.1]. Then, since the K-type of Φ° is fixed, according to the uniqueness of the newvector in the non-Archimedean case and the highest weight vector in the non-Archimedean case, we have

$$\mathcal{Z}\left(s+\frac{1}{2}, \Phi^\circ, \beta(\varphi_2, \varphi_1^\vee)\right) \neq 0$$

if and only if $\varphi_1 = \varphi_2 = \varphi^\circ$. Moreover, in this case

$$\mathcal{Z}\left(s+\frac{1}{2}, \Phi^\circ, \beta(\varphi^\circ, \varphi^\circ)\right) = L\left(s+\frac{1}{2}, \pi, \mathrm{st}\right).$$

Then we get the analytic continuation of $L(s, \pi, \mathrm{st})$.

As for the functional equation, according to [GJ72, Theorem 3.3, Proposition 6.12], there is an epsilon factor $\epsilon(s, \pi)$ such that

$$\begin{aligned}\sum_{\varphi_1.\varphi_2 \in \mathcal{B}(\pi)} & \mathcal{Z}\left(-s+\frac{1}{2}, \widehat{\Phi^\circ}, \beta^\vee(\varphi_2, \varphi_1^\vee)\right) \mathbb{W}_{\varphi_1}^{\psi^{-1}}(\mathrm{I}_2) \mathbb{W}_{\varphi_2^\vee}^{\psi}(\mathrm{I}_2) \\ &= \mathbb{W}_{\varphi^\circ}^{\psi^{-1}}(\mathrm{I}_2) \mathbb{W}_{\varphi^{\circ,\vee}}^{\psi}(\mathrm{I}_2) \epsilon\left(s+\frac{1}{2}, \pi\right) L\left(-s+\frac{1}{2}, \pi^\vee, \mathrm{st}\right),\end{aligned}$$

we may get

$$L\left(s+\frac{1}{2}, \pi, \mathrm{st}\right) = \epsilon\left(s+\frac{1}{2}, \pi\right) L\left(-s+\frac{1}{2}, \pi, \mathrm{st}^\vee\right).$$

$\square$

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